# On The Number Of Star-Shaped Polygons And Polyhedra

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## Abstract

We show that the maximum number of strictly star-shaped polygons through a given set of $n$ points in the plane is $\Theta(n^4)$. Our proof is constructive, i.e. we supply a construction which yields the stated number of polygons. We further present lower and upper bounds for the case of unrestricted star-shaped polygons. Extending the subject into three dimensions, we give a tight bound of $\Theta(n^9)$ on the number of distinct sets of star-shaped polyhedra.

## 1 Introduction

Reconstructing an initial geometric object from partial features has always been of major interest in computational geometry. Many well-studied problems in two and three dimensions can be seen from this point of view, for example triangulating a simple polygon [1] (or tetrahedralizing a polyhedron [6], respectively), finding the Delaunay triangulation of a point set, reconstructing three-dimensional objects from cross-sectional slices [2], or finding a triangulation of a three-dimensional polygon [5].

Instead of actually reconstructing an object it is an obvious related problem to find the maximum number of ways to reconstruct it. Euler and Goldbach, for example, showed an exact value of $\frac{1}{n-1}\binom{2n-4}{n-2}$ (the $(n-2)$th Catalan number, in fact) for the number of distinct triangulations of a convex $n$-gon (see, e.g. [7, 8]). Our paper deals with the maximum number of star-shaped polygons and polyhedra which connect a given set $S$ of points.

A geometric object is called *star-shaped*, if its boundary is entirely visible from an interior point or from a point on the boundary. By extension, we call an object *strictly star-shaped*, if this is valid even when visibility is restricted to the interior of the object. The number of distinct strictly star-shaped polygons through a fixed set of $n$ points was bound by $O(n^4)$ independently by Bieri and Schmidt [4] and by Auer and Held [3]. They showed that the number of polygons can not exceed the size of the partition induced by the $O(n^2)$ lines connecting each pair of points. Clearly, the size of such a partition is $O(n^4)$. However, they failed to provide a point arrangement that gives rise to that number of distinct polygons, leaving a gap which we will close in this paper. We will further prove asymptotic lower and upper bounds for the case of unrestricted star-shaped polygons.

The number of distinct polyhedra that can be drawn through a set of $n$ points in 3-space has not been explored before. We will see that with a straightforward generalization one can obtain similar results as in the two-dimensional case. Specifically, we will prove a tight worst-case bound of $\Theta(n^9)$ on the number of distinct sets of star-shaped polyhedra. The paper is structured as follows: The next chapter deals with strictly star-shaped polygons, whereas chapter 3 addresses unrestricted star-shaped polygons. Chapter 4 covers star-shaped polyhedra. The last chapter closes the paper with some related open problems.

## 2 Strictly Star-Shaped Polygons

The *kernel* of a star-shaped polygon is defined as the set of all points from which the polygon is fully visible. We denote by $S$ a set of $n$ points in the plane which will be the vertices of the polygons and we use $p$ to denote a point in the kernel of a specific polygon. Trivially, $p$ can only lie within the convex hull of $S$. Moreover, for the case of strict star-shapedness, $p$ can not lie collinearly with two (or more) points of $S$, since the closest point would shadow all further points. We start by establishing an upper bound on the number of strictly star-shaped polygons:

**Lemma 2.1** *The maximum number of distinct strictly star-shaped polygons through a set of $n$ points is $O(n^4)$.*

**Proof:** Every point $p$ inside the convex hull of $S$ not collinear with any line through two points of $S$ induces exactly one star-shaped polygon, whose vertices appear in sorted order around $p$. Such a polygon does not change as long as $p$ is not moved across any line through a pair of points in $S$. We therefore obtain an upper bound on the number of polygons by the arrangement of all lines through the points of $S$. Since the number of lines through $n$ points is $O(n^2)$, the size of the arrangement and thus the number of star-shaped polygons is $O(n^4)$. □

We will now describe a point set $S$ that allows to draw the maximum number of $O(n^4)$ distinct polygons. The set

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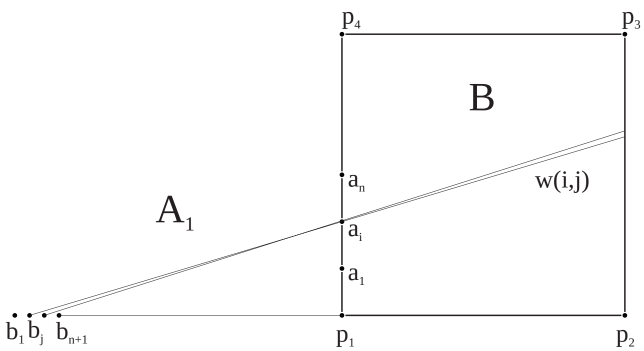


Figure 1: Point set construction for the proof of lemma 2.2: Auxiliary points $p_1$ to $p_4$ and point set $A_1$.

contains four auxiliary points $p_1$ to $p_4$ and two congruent point sets $A_1$ and $A_2$, with $2n+1$ points each. (Compare Figure 1.)

The four auxiliary points lie on the corners of a square centered at $(n, n)$ with edge length $2n$. The points are numbered in counter-clockwise order with $p_1$ at the origin. We will denote the square by $B$. The point set $A_1$ consists of a row $a$ of $n$ points along the y axis and a row $b$ of $n+1$ points on the negative x axis. The points $a_i$ (with $1 \le i \le n$) have coordinates $(0, i)$; the points $b_i$ (with $1 \le i \le n+1$) are uniformly distributed between $(-2n-1, 0)$ and $(-2n, 0)$, with $b_1$ at the left-most position (i.e. the coordinates of $b_i$ are $(-2n - \frac{i-1}{n}, 0)$). The point set $A_2$ is congruent to $A_1$, but is rotated by -90 degrees around the centre of $B$. (Compare Figure 2.)

**Lemma 2.2** *The point set $S$ described above gives rise to $\Omega(n^4)$ distinct strictly star-shaped polygons.*

**Proof:** Let $l(p, q)$ denote the line through $p$ and $q$. We define the wedge $w(i, j)$ with $1 \le i, j \le n$ as the set of points which lie both strictly above $l(a_i, b_j)$ and strictly below $l(a_i, b_{j+1})$ (refer to Figure 1). The spacing between the points of the sets $a$ and $b$ guarantees that the lines $l(b_j, a_i)$ do not intersect inside $B$. Therefore, all $n^2$ wedges $w(i, j)$ are mutually disjoint.

Let us first ignore the presence of the point set $A_2$. With the remaining point set, each point $p \in w(i, j) \cap B$ induces a polygon with the vertex order

$$(p_1 p_2 p_3 p_4 a_n \ldots a_{i+1} b_1 \ldots b_j a_i b_{j+1} \ldots b_{n+1} a_{i-1} \ldots a_1).$$

It takes a moment's time to verify that this vertex order is unique for each wedge. Excluding now $A_1$ from $S$, it follows from symmetry reasons and from the congruence of $A_2$ and $A_1$ that every point in each of $A_2$'s wedges will in turn induce a polygon which is unique to this wedge. The wedges of $A_1$ and $A_2$ intersect each other fully, producing $n^4$ quadrilaterals inside the square B. Each point inside a quadrilateral induces a unique polygon as the combination of the polygons obtained with the absence of $A_1$ or $A_2$, respectively. □

Combining Lemmas 2.1 and 2.2 we obtain the following theorem:

**Theorem 2.3** *The maximum number of distinct strictly star-shaped polygons through a set of $n$ points is $\Theta(n^4)$.*

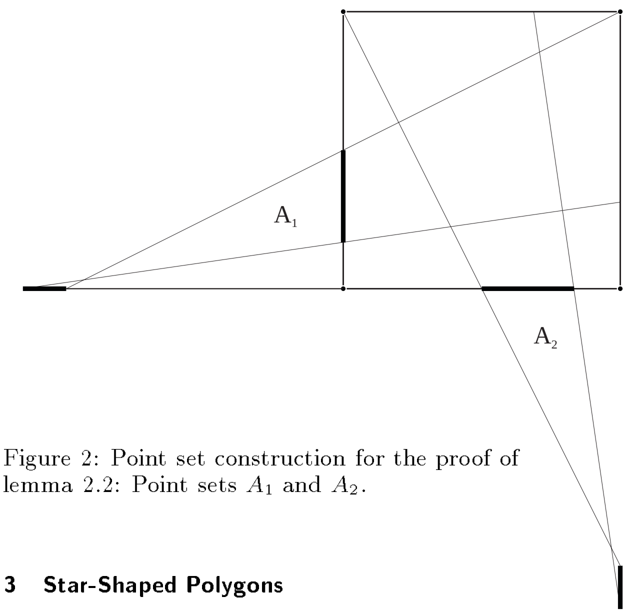


Figure 2: Point set construction for the proof of lemma 2.2: Point sets $A_1$ and $A_2$.

## 3 Star-Shaped Polygons

The key difference to strictly star-shaped polygons lies in the fact that a polygon center $p$ can now lie collinearly to arbitrarily many points of $S$. Such incidences vastly increase the number of possible polygons because every incidence allows two different ways to draw the polygon (towards or outwards $p$, as we shall see below). Before we state an upper bound on the maximum number of polygons we treat the polygon center as fixed and prove the following lemma:

**Lemma 3.1** *The maximum number of star-shaped polygons through a set of $n$ points and containing a fixed point $p$ is $(n-1)2^{\frac{n-7}{2}}$ for $n$ odd and $(n+4)2^{\frac{n-8}{2}}$ for $n$ even.*

**Proof:** We distinguish the four possible locations of $p$ with regard to the line arrangent of all lines through $S$ and take the maximum of all cases. If $p$ lies inside a partition cell, the number of polygons is exactly one, as shown above. In all the following cases we can arrange some points to lie collinearly to $p$. Whenever a ray emanating from $p$ crosses two or more points, we have the choice between drawing the polygon segment towards $p$, starting with the furthest point, or from $p$ away, starting with the closest point. Thus, each such incidence will double the number of polygons. All the following cases are depicted in Figure 2. With $p$ coinciding with a line, but not lying on a line intersection, at most two such incidences can occur, leading to a maximum number of polygons of four. If $p$ lies on a line intersection disjunct from $S$, at most $n/2$ such incindences can occur, which bounds the maximum number by $2^{n/2}$. Finally, if $p \in S$, we can - with an odd number of $n$ - arrange $n-1$ points to be pairwise collinear to $p$. In addition, we have the choice of which points on two subsequent rays connect to $p$. This leads to a maximum of $\frac{n-1}{2} 2^{\frac{n-5}{2}} = (n-1)2^{\frac{n-7}{2}}$ for the number of polygons, which in turn is the maximum over all four cases. For an even number of $n$, a similar calculation leads to $(n+4)2^{\frac{n-8}{2}}$ for the number of polygons. □

Next, we will treat the polygon centers as movable and obtain the following upper bound:

**Lemma 3.2** *The maximum number of distinct star-shaped polygons through a set of $n$ points is $O(n^4 2^{n/2})$ and $\Omega(n 2^{n/2})$.*

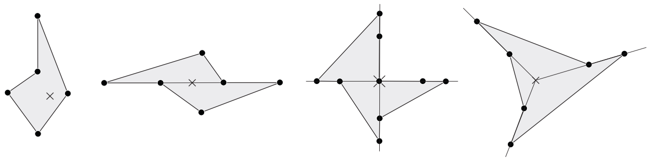

Figure 3: Worst-case point arrangements for the proofs of Lemmas 3.1 and 3.2

**Proof:** We exploit the results in the proof of Lemma 3. For the upper bound we again distinguish the four locations of a polygon center $p$ as stated above and now multiply the individual maximum numbers of polygons with the maximum number of such locations. Clearly, the number of partition cells, cell edges and cell corners are all $O(n^4)$, while by definition the size of $S$ is $n$. The sum of the four products gives the desired upper bound.

The point arrangement described above for the case where $p \in S$ allowed $(n-1)2^{\frac{n-7}{2}}$ distinct polygons. This number is $\in \Omega(n2^{n/2})$. □

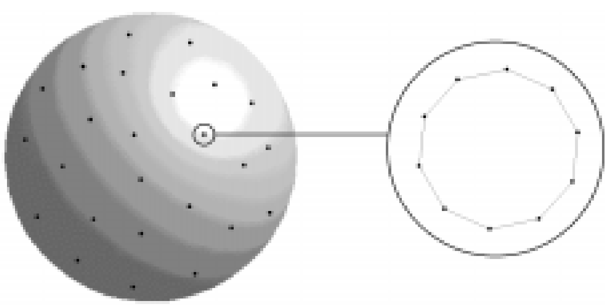

Figure 4: Point arrangement for the proof of lemma 4.1

## 4 Star-Shaped Polyhedra

In the two-dimensional case we have observed a reduction from an exponential to a polynomial number of polygons with the introduction of strict star-shapedness. The following lemma states that we can not expect the same behavior in three dimensions.

**Lemma 4.1** *The number of distinct strictly star-shaped polyhedra through a set of n points is $\Omega(2^n)$.*

**Proof:** We speckle the surface of a sphere with $n/10$ points and split up each point into ten new points which we place on the corners of arbitrarily small decagons, such that all points remain on the surface of the sphere. (Compare Figure 4.) This final arrangement consists of $n$ points and $n/10$ decagons. Each decagon can be independently triangulated in 1430 different ways (see, e.g. [7, 8]), which leads to at least $1430^{n/10}$ distinct ways to triangulate the whole polyhedron. With $2^{10} < 1430$ we have $1430^{n/10} \notin O(2^n)$, hence the number of polyhedra must be strictly more than $\Omega(2^n)$. □

Since, as the above point set shows, a fixed point in a kernel can give rise to a set of possible star-shaped polyhedra, it is a straightforward generalization of the two-dimensional

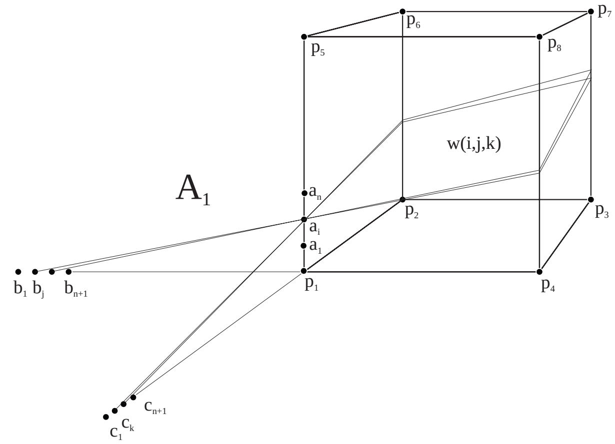


Figure 5: Point set construction for the proof of lemma 4.3: Auxiliary points $p_1$ to $p_8$ and point set $A_1$.

case to ask about the maximum number of such distinct sets. Let the function $f$ assign to each point $p$ inside the convex hull of $S$ the set of all star-shaped polyhedra through $S$ which are visible from $p$. We consider a partition $P$ of the convex hull of $S$ into cells such that two points $p_1$ and $p_2$ belong to the same partition cell if $f(p_1) = f(p_2)$. We first establish an upper bound on the size of this partition:

**Lemma 4.2** *The size of the partition $P$ is $O(n^9)$.*

**Proof:** The proof closely follows the proof of lemma 2.1. The crucial observation is that the set $f(p)$ of polygons can only change when $p$ moves across a plane through three points of $S$. Hence the size of the arrangement of all planes through any three points of $S$ gives an upper bound on the size of the partition. With $n$ points in $S$ we obtain a maximum of $O(n^3)$ planes with an arrangement size of at most $O(n^9)$. This in turn leads to the desired partition size of $O(n^9)$. □

Extending the results from chapter 2, we will describe a point set whose induced partition size indeed is $O(n^9)$. The set consists of eight auxiliary points $p_1$ to $p_8$ and three congruent point sets $A_1$ to $A_3$, containing $3n+2$ points each.

The eight auxiliary points lie on the corners of a cube centered at $(n, n, n)$ with edge length $2n$. We will refer to this cube as $C$. The point set $A_1$ consists of three rows $a$, $b$, and $c$ of collinear, uniformly distributed points, positioned on the y-, x-, and z-axis, respectively. The points of row $a$ have coordinates $(0, i, 0)$, with $i$ from 1 to $n$. The points of row $b$ spread uniformly from $(-2n-1, 0, 0)$ to $(-2n, 0, 0)$, such that $b_i$ has coordinates $(-2n - \frac{i-1}{n}, 0, 0)$, with $i$ from 1 to $n+1$. Row $c$ finally has its points uniformly spread from $(0, 0, -2n - \frac{1}{n})$ to $(0, 0, -2n)$, thus $c_i$ has coordinates $(-2n - \frac{i-1}{n^2}, 0, 0)$, with $i$ again ranging from 1 to $n+1$. The point set $A_2(A_3)$ is a congruent copy of $A_1$ after a rotation by -90 degrees around an x(z)-axis through the centre of $C$. (Compare Figures 5 and 6.)

**Lemma 4.3** *The point set $S$ described above gives rise to $\Omega(n^9)$ distinct sets of star-shaped polyhedra.*

**Proof:** Let $p(a, b, c)$ denote the plane through three points $a$, $b$, and $c$. We define the wedge $w(i, j, k)$ with $1 \leq i, j, k \leq n$ as the set of points which lie both strictly above $p(a_i, b_j, c_i)$ and strictly below $p(a_i, b_{j+1}, c_{j+1})$ (refer to Figure 5). The spacing between the point of sets $a$, $b$ and $c$ was chosen such

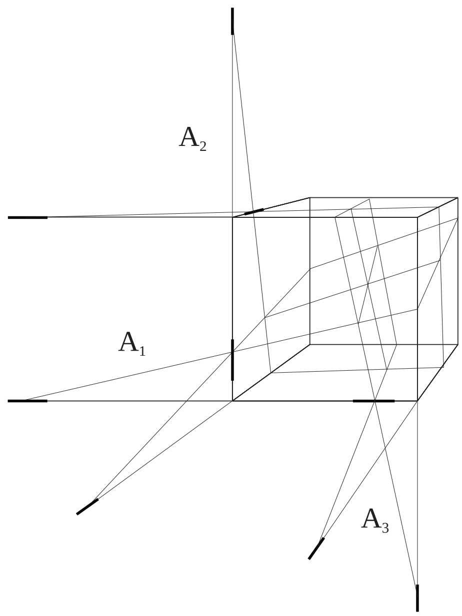


Figure 6: Point set construction for the proof of lemma 4.3: Point sets $A_1$ to $A_3$.

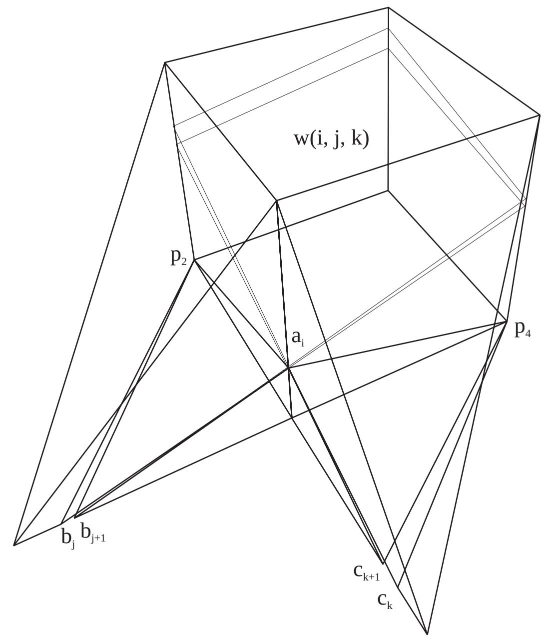


Figure 7: A star-shaped polyhedra specific to a point $p \in w(i,j,k)$

that the planes $p(a_i, b_j, c_k)$ do not intersect inside $C$. Therefore, all $n^3$ wedges $w(i,j,k)$ are mutually disjoint inside the cube.

Let us first disregard the presence of the point sets $A_2$ and $A_3$. With the remaining point set, each point $p \in w(i,j,k) \cap C$ induces a set of polyhedra which, amongst others, contains polyhedra as depicted in Figure 7. All these polyhedra share as their key feature the triangle pairs $(p_2, b_j, a_i)/(p_2, b_{j+1}, a_i)$ and $(p_4, c_k, a_i)/(p_4, c_{k+1}, a_i)$. It is easy to see that no other set $f(p')$ with $p' \in w(i', j', k')$ can contain polyhedra of this kind since at least one of the characteristic triangles would shadow a point in $S$, invalidating the star-shapedness property.

Excluding now $A_1$ and $A_3$ from $S$, it follows from symmetry reasons and from the congruence of $A_1$, $A_2$, and $A_3$ that every point in each of $A_2$'s wedges will in turn induce a set of polyhedra which contains members that are unique to the chosen wedge. Analogously, the same applies if we exclude the sets $A_1$ and $A_2$. The wedges of $A_1$, $A_2$, and $A_3$ mutually fully intersect, producing $n^9$ hexahedra inside the cube $C$. Let us now choose an arbitrary point $p$ inside such a hexahedron. The induced set of polyhedra $f(p)$ contains members which are a combination of the characteristic polyhedra obtained with the isolated treatment of $A_1$, $A_2$, or $A_3$, respectively. These members can not feature in any other set of polyhedra, leading to $\Omega(n^9)$ distinct such sets. □

Combining Lemmas 4.2 and 4.3 we obtain the following theorem:

**Theorem 4.4** *The maximum number of distinct sets of star-shaped polyhedra through a set of n points is* $\Theta(n^9)$.

## 5 Open Problems

The scope of this paper can be extended in various directions. First, it would be straightforward to generalize the results to star-shaped polytopes of arbitrary dimensions. We expect that the number of distinct sets of $d$-dimensional star-shaped polytopes is $\Theta(n^{d^2})$. With a view to Combinatorial Geometry, instead of contenting with asymptotic bounds, it would be interesting to explore the exact maximum of the number of (strictly) star-shaped polytopes. I am not aware of any research in this field. There is an unsatisfactorily large gap between the lower and the upper bound of the number of star-shaped polygons. It would be very nice to bring them together as could be done in the case of strict star-shapedness.

Dropping the requirement of star-shapedness, the maximum number of simple polytopes through a set of $n$ points is still an open question. For the two-dimensional case, it is not even known whether there exists a polynomial-time algorithm that generates a random simple polygon with a uniform distribution [4].

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