On two lower bound constructions

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Abstract

We address two problems. In the first, we refine the analysis of a lower bound construction of a point set having many non-crossing spanning trees. We also give more precise upper bounds on the maximum number of non-crossing spanning trees, perfect matchings and simple polygons of a planar point set. In the second, we give an improved lower bound construction for the d-interval problem.

1 Non-crossing subgraphs – an introduction

Consider a set of n points in the plane and the straight-line drawing of the complete graph K_n they define. A subgraph of K_n in this drawing is called non-crossing (or crossing-free) if its edges intersect only at common vertices. Ajtai, Chvátal, Newborn and Szemerédi proved that the number of non-crossing subgraphs of any drawing of K_n (even without the rectilinear restriction on edges) is bounded from above by 10^{13n}. This result was a consequence of a lower bound on the crossing number of a graph G (i.e. the minimum number of crossing pairs of edges over all planar drawings of G). An improvement in this lower bound on crossing number [PT97] lead to an improved upper bound of 53000^n on the number of non-crossing subgraphs. Further improvements on the number of non-crossing subgraphs have been obtained by bounding the number of triangulations of a planar point set. As noted in [GNT95], since every non-crossing subgraph can be extended to a triangulation, and since a triangulation has at most $3n$ edges, a bound of α^n on the number of triangulations implies a bound of $2^{3n}\alpha^n = (8\alpha)^n$ on the number of non-crossing subgraphs. Smith [S89] proved a bound of 173000^n on the number of triangulations of any planar point set with n points. An improved bound (on the number of triangulations) of $2^{12.245113n-\Theta(\log n)} \leq 4855^n$ was found by Seidel [S99], which was further improved by Denny and Sohler [DS97]. They proved a bound of $2^{8.12n+O(\log n)} \leq 279^n$.

On the other hand, lower bounds on the maximum number of non-crossing subgraphs are provided by specific configurations of points (see [GNT95] for a review of results and for the latest improvements at this time, that we are aware of). First [A79], then Hayward [H87], later García and Tejel [GT99] gave lower bounds of 2.27^n, 3.26^n and 3.605^n respectively on the number of simple polygons. The best bounds we know are from [GNT95]: 4.642^n for simple polygons, $\Omega(8^n/n^{O(1)}) = \Omega((8-\epsilon)^n)$ for triangulations, $\Omega(3^n/n^{O(1)}) = \Omega((3-\epsilon)^n)$ for perfect matchings and $\Omega(9.35^n)$ for spanning trees ($\epsilon > 0$ is arbitrarily small).

Here we improve the analysis of a construction given in [GNT95], obtaining a better lower bound on the number of non-crossing spanning trees. We also get more precise upper bounds for these three types of subgraphs (polygons, matchings and trees) as derived from an upper bound on the number of triangulations.

2 A lower bound for spanning trees

We give a sharper analysis of a configuration S of n points given in [GNT95]. The parameters α, β, γ are to be specified later. The points of S are partitioned into two convex chains with opposite concavity (see Figure 1), $|S_1| = \alpha n$ points on the upper chain C_1 and $|S_2| = (1-\alpha)n$ points on the lower chain C_2. Take any partition $S_1 = T_1 \cup F_1$ of the points in C_1 with $|T_1| = (\alpha - \beta)n,\ |F_1| = \beta n$. Select any subset M_1 from T_1 with $|M_1| = \gamma n$.

The points (vertices) in T_1, F_1, M_1 are called tree points, free points and matched points, respectively. Matched points are labeled by "m", (the rest of tree vertices are unlabeled) and the free points are labeled by "f" as in Figure 1. Take any spanning tree on the tree

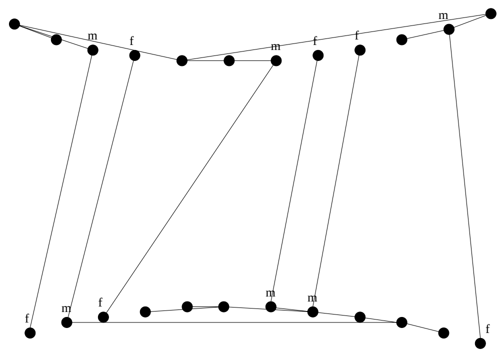

Figure 1: Counting spanning trees in S

vertices of T_1. Choose any partition $S_2 = A_2 \cup B_2$ of the points in C_2 with $|A_2| = (\beta+\gamma)n,\ |B_2| = (1-\alpha-\beta-\gamma)n$. Scan the points in $M_1 \cup F_1$ from left to right and partition the vertices in A_2 into two sets, matched vertices M_2 and free vertices F_2 of S_2 in the following way: if the i-th point in $M_1 \cup F_1$ is a matched point, make the corresponding point in A_2 a free point, and vice versa. Construct the set of tree vertices T_2 in S_2 as $T_2 = M_2 \cup B_2$. Take any spanning tree on the tree vertices of T_2. Match vertices of M_1 with vertices of F_2 and vertices of M_2 with vertices of F_1 when scanning $M_1 \cup F_1$ and $M_2 \cup F_2$, one point from each set at a time, from left to right. Finally, add an extra edge (say in S_2) to produce a spanning tree of S. It is known ([M48], [DP93]) that the number of non-crossing spanning trees of n points in convex position is given by

$$t_n = \frac{1}{2n-1}\binom{3n-3}{n-1} =$$
$$= \Theta(n^{-\frac{3}{2}}2^{3H(\frac{1}{3})n}) = \Theta(n^{-\frac{3}{2}}(27/4)^n)$$

The following is a lower bound on the number of spanning trees we obtain.

$$T_n \geq \binom{\alpha n}{\beta n}\binom{(\alpha-\beta)n}{\gamma n} t_{(\alpha-\beta)n} \binom{(1-\alpha)n}{(\beta+\gamma)n} t_{(1-\alpha-\gamma)n}$$

Denote by $H(q) = -q\log q - (1-q)\log(1-q)$ the binary entropy function, where log denotes the logarithm in base 2. From the well-known estimate

$$\binom{n}{\alpha n} = \Theta(n^{-\frac{1}{2}}2^{H(\alpha)n})$$

we get that

$$\binom{\alpha n}{\beta n} = \Theta(n^{-\frac{1}{2}}2^{\alpha H(\frac{\beta}{\alpha})n})$$

$$\binom{(\alpha-\beta)n}{\gamma n} = \Theta(n^{-\frac{1}{2}}2^{(\alpha-\beta)H(\frac{\gamma}{\alpha-\beta})n})$$

$$\binom{(1-\alpha)n}{(\beta+\gamma)n} = \Theta(n^{-\frac{1}{2}}2^{(1-\alpha)H(\frac{\beta+\gamma}{1-\alpha})n})$$

The inverse polynomial factors can be ignored without affecting the final result (from the strict inequality on the function $E(\cdot)$ further below).

$$t_{(\alpha-\beta)n}t_{(1-\alpha-\gamma)n} = \Omega(2^{3H(\frac{1}{3})(1-\beta-\gamma)n})$$

Finally,

$$T_n \geq \Omega(2^{E(\alpha,\beta,\gamma)n}), \quad \text{where}$$

$$E(\alpha,\beta,\gamma) = \alpha H(\frac{\beta}{\alpha}) + (\alpha-\beta)H(\frac{\gamma}{(\alpha-\beta)}) +$$
$$+ (1-\alpha)H(\frac{\beta+\gamma}{1-\alpha}) + 3H(\frac{1}{3})(1-\beta-\gamma)$$

It can be checked that $E(0.5, 0.088, 0.088) > 3.3819$ which gives the bound

$$T_n = \Omega(2^{3.3819n}) = \Omega(10.42^n)$$

2.1 Specific upper bounds

As noted in the Introduction, if we have an upper bound of α^n on the number of triangulations, we obtain from it a bound of $(8\alpha)^n$ on the total number of non-crossing subgraphs. However, for certain classes of subgraphs better bounds are derivable from it.

For the case of perfect matchings, since any triangulation has at most $3n$ edges and any perfect matching has $n/2$ edges, their number is bounded as follows

$$M_n \leq 2^{3H(\frac{1}{6})n}\alpha^n = (2^{3H(\frac{1}{6})}\alpha)^n \leq (3.87\alpha)^n$$

Here we have used the estimate

$$\binom{3n}{\frac{n}{2}} = \Theta(n^{-\frac{1}{2}}2^{3H(\frac{1}{6})n})$$

Similarly, since every simple polygon has n edges, their number is bounded as

$$H_n \leq 2^{3H(\frac{1}{3})n}\alpha^n = (2^{3H(\frac{1}{3})}\alpha)^n = (6.75\alpha)^n$$

Finally, since every spanning tree has $n-1$ edges, their number is bounded by the same quantity

$$T_n \leq 2^{3H(\frac{1}{3})n}\alpha^n = (2^{3H(\frac{1}{3})}\alpha)^n = (6.75\alpha)^n$$

Remark. Let $M(S)$ stand for the number of non-crossing matchings of a finite planar point set S using straight lines. Similarly, denote by $P(S), U(S), T(S)$ the number of simple polygons, triangulations and spanning trees, respectively. It would be interesting to decide if there are any of these quantities always in the same order.

3 d-intervals – an introduction

For a positive integer d, a (homogeneous) d-interval is a union of d closed intervals on a (same) line. Let $\mathcal{H}$ be a finite collection of d-intervals. The matching number of $\mathcal{H}$, $\nu(\mathcal{H})$, is the maximum number of pairwise disjoint elements of $\mathcal{H}$. The transversal number of $\mathcal{H}$, $\tau(\mathcal{H})$, is the minimum number of points that intersect every member of $\mathcal{H}$. For any hypergraph $\mathcal{H}'$, $\nu(\mathcal{H}') \leq \tau(\mathcal{H}')$ is a trivial inequality. Gyárfás and Lehel [GL70] proved that $\tau \leq O(\nu^{d!})$ and Kaiser [K97] proved that $\tau \leq O(d^2\nu)$. More exactly, the latter bound is $(d^2 - d + 1)\nu$ in general, and $(d^2 - d)\nu$ for certain values of $d \geq 3$. Recently Alon [A98] has given a simpler proof of a slightly weaker bound: $\tau \leq 2d^2\nu$.

In [K97] it is pointed as an open problem to improve the existing lower bound of possible transversal number of systems of d-intervals. As mentioned there and also to our knowledge, the best available is $\tau \geq d\nu$ for general d. In the special case $d = 2$, the best upper bound $\tau \leq 3\nu$ is tight. This follows from a construction taken from [GL70], which appears also in [K97], of a family of 2-intervals having $\nu = 1$ and $\tau = 3$. It is shown in Figure 2. Using several disjoint copies of this family, one can see that the above inequality is optimal for all ν. To our knowledge, the case $d = 2$ is the only one for which exact bounds are known. In the next Section, we generalize this construction for any d, proving a lower bound of $(2d - 1)\nu$.

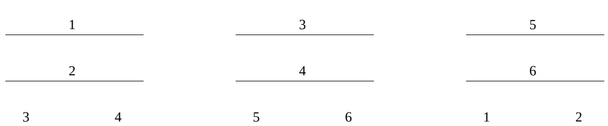

Figure 2: A family of 2-intervals with $\nu = 1$ and $\tau = 3$

4 Construction of a system of d-intervals

We use the following notation. For two positive integers, $i < j$, $[i, j]$ means the list of all integers between i and j: $\{i, i + 1, \dots, j\}$. Given three lists of integers, $l1, l2, l3$, we say that $C(l1; l2, l3)$ is a column of intervals, if
(i) all intervals identified by elements of $l1$ have the same length and overlap; these are called long intervals.
(ii) the intervals identified by elements of $l2$ and $l3$ have an equal but smaller length, are disjoint, follow the order in the concatenation of the two lists and their union is included in any long interval of $l1$; these are called short intervals.

Using this notation, the system of 6 2-intervals in Figure 2 is described by

$$C([1, 2]; [3, 4]), \quad C([3, 4]; [5, 6]), \quad C([5, 6]; [1, 2])$$

In Figure 3, it is shown a column of intervals $C([28, 36]; [37, 45], [1, 9]$ with 9 long and 18 short intervals. In a similar way, we can have columns defined by only two lists $C(l1; l2)$, in which case the short intervals are identified by one list only.

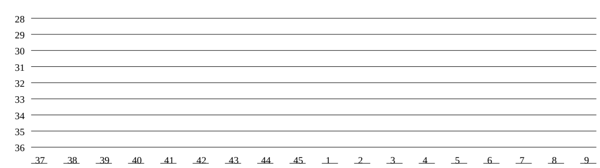

Figure 3: A column of intervals: $C([28, 36]; [37, 45], [1, 9]$

Let $x = 4d - 3$. Our system of d-intervals consists of several disjoint columns, linearly ordered from left to right:

$$\begin{aligned}
&C([1, x]; [x + 1, dx]),\\
&C([x + 1, 2x]; [2x + 1, (d + 1)x]),\\
&\vdots\\
&C([(d - 1)x + 1, dx]; [dx + 1, (2d - 1)x]),\\
&C([dx + 1, (d + 1)x]; [(d + 1)x + 1, (2d - 1)x], [1, x]),\\
&C([(d + 1)x + 1, (d + 2)x]; [(d + 2)x + 1, (2d - 1)x],\\
&\qquad [1, 2x]),\\
&\vdots\\
&C([(2d - 2)x + 1, (2d - 1)x]; [1, (d - 1)x])
\end{aligned}$$

Write c for the number of columns, $c = 2d-1$. It can be checked that for this system of d-intervals, $\nu = 1$. If D is the maximum degree of the hypergraph (the maximum number of d-intervals containing any point), $D = x+1 = 4d-2$. The total number of d-intervals is $m = (2d-1)x = (2d-1)(4d-3)$. Clearly,

$$\tau \geq \frac{m}{D} = \frac{(2d-1)(4d-3)}{(4d-2)} = \frac{4d-3}{2} = 2d - 1.5$$

This implies $\tau \geq 2d-1$. It can also be seen that $\tau \leq c = 2d-1$, hence $\tau = 2d-1$.

By taking several disjoint copies of this system, for any ν, one can get systems for which $\tau = (2d-1)\nu$.

For small values of d, we obtain the following systems of d-intervals.

$$d=2: \quad C([1,5];[6,10]), \quad C([6,10];[11,15]),$$
$$C([11,15];[1,5])$$

$$d=3: \quad C([1,9];[10,27]), \quad C([10,18];[19,36]),$$
$$C([19,27];[28,45]), \quad C([28,36];[37,45],[1,9]),$$
$$C([37,45];[1,18])$$

(the 4-th column in this system appears in Figure 3)

$$d=4: \quad C([1,13];[14,52]), \quad C([14,26];[27,65]),$$
$$C([27,39];[40,78]), \quad C([40,52];[53,91]),$$
$$C([53,65];[66,91],[1,13]),$$
$$C([66,78];[79,91],[1,26]), \quad C([79,91];[1,39])$$

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