A note on the path graph of a set of points in convex position in the plane

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Extended abstract

1 Introduction

For any connected abstract graph G, the *tree graph* $T(G)$ is the graph that has one vertex for each spanning tree of G and an edge joining trees R and S whenever R is obtained from S by a single edge exchange. R. L. Cummings proved in [C] that $T(G)$ is hamiltonian; see also [S] for a short proof.

A geometric variation that has been studied is the following: For a set P of points in general position in the plane the *plane tree graph* $T(P)$ of P is defined as the abstract graph with one vertex for each plane spanning tree of P, in which two trees are adjacent if, as in the abstract case, one is obtained from the other by a single edge exchange. D. Avis and K. Fukuda proved in [A] that $G(P)$ is always connected. In [H], C. Hernando *et al* show that if the points in P are the vertices of a convex polygon, then $G(P)$ is hamiltonian.

In this note we only consider sets P of points in convex position and study the subgraph $G(P)$ of $T(P)$, induced by the set of plane spanning paths of P. We prove that $G(P)$ is itself hamiltonian.

Since for any spanning path T of P planarity depends only on the relative position of its vertices along the convex hull of P, then for any set P of n points in convex position in the plane, the graph $G(P)$ is isomorphic to $G(P_n)$, where P_n is a regular n-gon. We denote by G_n the graph $G(P_n)$. The graphs G_3 and G_4 are shown in Figure 1.

The main result of this article is the following.

Theorem 1. *If $n \geq 3$, then G_n is hamiltonian.*

Throughout the paper, $w_1, w_2, \ldots, w_n$ denote the vertices of P_n in clockwise order. Addition of integers is taken modulo n.

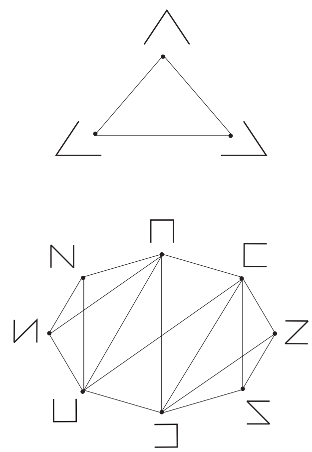

Figure 1.- The graphs G_3 and G_4

2 Preliminary results

A natural partition of the set of plane spanning paths of P_{2m+1} into $2m+1$ sets $A_1, A_2, \ldots, A_{2m+1}$ is as follows: A path T is in A_t if and only if the middle point of T is w_t. In this section, we prove that the subgraph of G_{2m+1}, induced by A_{m+1} contains a particular Hamilton path which will be useful in the prove of Theorem 1.

Let $n = 2m+1$ and for $i = 1, 2, \ldots, m+1$ let $u_i = w_i$ and $v_i = w_{2m-i+2}$. Any path $T \in A_{m+1}$ consists of a left subpath T_L with one end in $u_{m+1} = w_{m+1}$ and vertex set $U_{m+1} = \{u_1, u_2, \ldots, u_m, u_{m+1}\}$ and

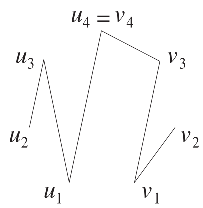

Figure 2.- $L = u_2, u_3, u_1, u_4$ and $R = v_4, v_3, v_1, v_2$

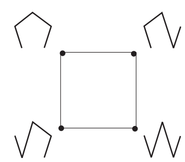

Figure 3.- The graph F_3

a right subpath T_R with one end in $v_{m+1} = w_{m+1}$ and vertex set $V_{m+1} = \{v_1, v_2, \ldots, v_m, v_{m+1}\}$. For any plane paths L and R with one end in w_{m+1} and vertex sets U_{m+1} and V_{m+1}, respectively, we denote by $L * R$ the path in A_{m+1} with left subpath L and right subpath R (see Figure 2).

Let $\theta : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$ given by $\theta(k) = m + 1 - k$. For any plane path L with vertex set U_{m+1} and with one end in u_{m+1} let $\theta(L - u_{m+1})$ be the plane path, with vertex set $U_m = \{u_1, u_2, \ldots, u_m\}$, in which $u_{\theta(t)}$ and $u_{\theta(s)}$ are adjacent if and only if u_t and u_s are adjacent in L. A path $\theta(R - u_{m+1})$, with vertex set $V_m = \{v_1, v_1, \ldots, v_m\}$, is defined in an analogous way for any plane path R with one end in v_{m+1} and with vertex set V.

Let F_{m+1} denote the subgraph of G_{2m+1}, induced by A_{m+1}, and for $t = 1, 2, \ldots, m$, let $L_t = u_1, u_2, \ldots, u_t, u_{t+1}$, $R_t = u_{t+1}, v_t, \ldots, v_1$, $L'_t = u_t, u_{t-1}, \ldots, u_1, u_{t+1}$ and $R'_t = v_{t+1}, v_1, v_2, \ldots, v_t$.

Theorem 2. *If $m \geq 2$, then F_{m+1} contains a Hamilton path J_{m+1} with ends $L_m * R_m$ and $L_m * R'_m$ and a Hamilton path J'_{m+1} with ends $L_m * R_m$ and $L'_m * R_m$.*

Proof. Figure 3 shows the graph of F_3. We proceed by induction assuming $m \geq 3$ and that the result holds for $m' = m - 1$; by symmetry, we only need to show a Hamilton path in F_{m+1} with ends $L_m * R_m$ and $L_m * R'_m$. For $i, j \in \{1, m\}$ let $A^{i,j}_{m+1}$ be the set of paths in A_{m+1} containing the edges $u_i w_{m+1}$ and $w_{m+1} v_j$. We claim that the subgraph of G_{2m+1}, induced by $A^{i,j}_{m+1}$ is isomorphic to $F_m = F_{m'+1}$.

For $i, j \in \{1, m\}$ let $\alpha_{i,j} : A^{i,j}_{m+1} \to A_m$ given by $\alpha_{i,j}(T) = \theta^{\frac{m-i}{m-1}}(T_L - w_{m+1}) * \theta^{\frac{m-j}{m-1}}(T_R - w_{m+1})$; notice that $\theta^{\frac{m-1}{m-1}} = \theta^1 = \theta$ and $\theta^{\frac{m-m}{m-1}} = \theta^0 = I$ (identity function). Let $T \in A^{i,j}_{m+1}$; since $u_i w_{m+1}$ and $w_{m+1} v_j$ are edges of T, then $T_L - w_{m+1}$ has an end in u_i and $T_R - w_{m+1}$ has an end in v_j and since $\theta^{\frac{m-1}{m-1}}(1) = \theta(1) = m$ and $\theta^{\frac{m-m}{m-1}}(m) = I(m) = m$, then $\theta^{\frac{m-i}{m-1}}(T_L - w_{m+1})$ has an end in u_m and $\theta^{\frac{m-j}{m-1}}(T_R - w_{m+1})$ has an end in v_m. Therefore $\alpha_{i,j}(T) \in A_m$. Since θ preserves adjacency in the sense that for $s, t \in \{0, 1\}$, the paths $L * R$ and $M * S$ are adjacent in F_{m+1} if and only if $\theta^s(L - w_{m+1}) * \theta^t(R - w_{m+1})$ and $\theta^s(M - w_{m+1}) * \theta^t(S - w_{m+1})$ are adjacent in F_m, then for $i, j \in \{1, m\}$, two paths R and S in $A^{i,j}_{m+1}$ are adjacent in F_{m+1} if and only if $\alpha_{i,j}(R)$ and $\alpha_{i,j}(S)$ are adjacent in F_m.

By induction F_m contains a Hamilton path J_m with ends in $L_{m-1} * R_{m-1}$ and $L_{m-1} * R'_{m-1}$; therefore for $i, j \in \{1, m\}$, the subgraph of G_{2m+1}, induced by $A^{i,j}_{m+1}$ contains a Hamilton path $J^{i,j}_{m+1}$ with ends $\alpha^{-1}_{i,j}(L_{m-1} * R_{m-1})$ and $\alpha^{-1}_{i,j}\left(L_{m-1} * R'_{m-1}\right)$. To end the proof we show how to connect the paths $J^{m,m}_{m+1}$, $J^{1,m}_{m+1}$, $J^{1,1}_{m+1}$ and $J^{m,1}_{m+1}$ to form a Hamilton path J_{m+1} of F_{m+1} with ends $\alpha^{-1}_{m,m}(L_{m-1} * R_{m-1}) = L_m * R_m$ and $\alpha^{-1}_{m,1}\left(L_{m-1} * R'_{m-1}\right) = L_m * R'_m$.

The path $\alpha^{-1}_{1,m}\left(L_{m-1} * R'_{m-1}\right)$ can be obtained from $\alpha^{-1}_{m,m}\left(L_{m-1} * R'_{m-1}\right)$ by deleting the edge $u_m w_{m+1}$ and adding the edge $u_1 w_{m+1}$, therefore $\alpha^{-1}_{m,m}\left(L_{m-1} * R'_{m-1}\right)$ and $\alpha^{-1}_{1,m}\left(L_{m-1} * R'_{m-1}\right)$ are adjacent in F_{m+1}. Analogously $\alpha^{-1}_{1,m}(L_{m-1} * R_{m-1})$ and $\alpha^{-1}_{1,1}(L_{m-1} * R_{m-1})$ are adjacent in F_{m+1} and also $\alpha^{-1}_{1,1}\left(L_{m-1} * R'_{m-1}\right)$ and $\alpha^{-1}_{m,1}\left(L_{m-1} * R'_{m-1}\right)$ are adjacent in F_{m+1}. □

For $n = 2m$, let B_m be the set of plane spanning paths of P_{2m} with middle edge $w_i w_j$ ($i \in \{1, m\}$ and $j \in \{m+1, 2m\}$); with left subpath T_L with one end in w_i and vertex set $\{w_1, w_2, \ldots, w_m\}$ and right subpath T_R with one end in w_j and vertex set $\{w_{m+1}, w_{m+2}, \ldots, w_{2m}\}$. Let H_m be the subgraph of G_{2m}, induced by B_m (see Figure 4).

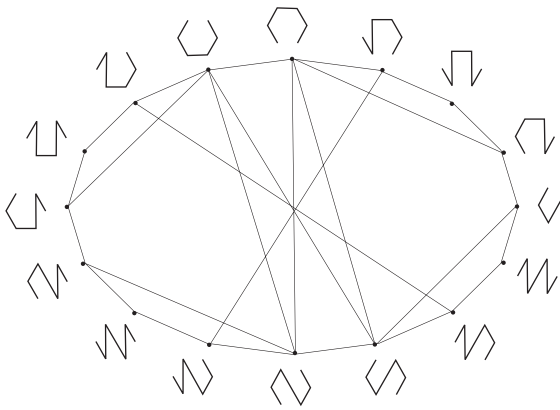

Figure 4.- The graph H_3

Let C be the cycle $w_1, w_2, \ldots, w_{2m}, w_1$ and T_m and T'_m be the paths $C - w_{2m}w_1$ and $C - w_m w_{m+1}$, respectively. The following theorem is presented here without a proof.

Theorem 3. *If $m \geq 2$, H_m, contains a Hamilton path with ends T_m and T'_m.*

3 Proof of Theorem 1

For $n \geq 5$ we consider two cases.
Case 1.- $n = 2m + 1$.

For $k = 0, 1, \ldots, 2m$ let A_{k+1} be the set of plane spanning paths of P_{2m+1} with middle point w_{k+1} and F_{k+1} be the subgraph of G_{2m+1}, induced by A_{k+1}. Let $\lambda : P_{2m+1} \rightarrow P_{2m+1}$ given by $\lambda(w_t) = w_{t+1}$. Since A_{k+1} is obtained from A_{m+1} by the rotation defined by λ^{k-m}, then F_{k+1} is isomorphic to F_{m+1}. By Theorem 2, F_{m+1} contains a Hamilton path J_{m+1} with ends $L_m * R_m$ and $L_m * R'_m$, therefore F_{k+1} contains a Hamilton path J_{k+1} with ends $\lambda^{k-m}(L_m * R_m)$ and $\lambda^{k-m}(L_m * R'_m)$. To end the proof we show how to connect $J_1, J_2, \ldots, J_{2m+1}$ to obtain a Hamilton cycle of G_{2m+1}.

Since $\lambda^{m+1}(L_m * R_m) = (L_m * R'_m - w_{m+1}w_{2m+1}) + w_{2m+1}w_1$, then $L_m * R'_m$ and $\lambda^{m+1}(L_m * R_m)$ are adjacent in G_{2m+1}. Therefore $\lambda^{t(m+1)}(L_m * R'_m)$ and $\lambda^{(t+1)(m+1)}(L_m * R_m)$ are adjacent in G_{2m+1} for $t = 0, 1, \ldots, 2m$. Since $0, m+1, 2(m+1), \ldots, 2m(m+1)$ are all the different residues modulo $2m + 1$ and $(2m+1)(m+1) \equiv 0 \mod (2m+1)$, then $J_{m+1}, J_{2(m+1)}, \ldots, J_{2m(m+1)}$ can be connected, in this order, to form a Hamilton cycle of G_{2m+1}.
Case 2.- $n = 2m$.

For $k = 1, 2, \ldots, m$ let B_k be the set of plane spanning paths of P_{2m} with middle edge $w_i w_j$ (with $i \in \{k, k-m+1\}$ and $j \in \{k+1, k+m\}$) and left subpath with vertex set $\{w_{k-m+1}, w_{k-m+2}, \ldots, w_k\}$ and right subpath with vertex set $\{w_{k+1}, w_{k+2}, \ldots, w_{k+m}\}$. Let H_k be the subgraph of G_{2m} induced by B_k

Let $\mu : P_{2m} \rightarrow P_{2m}$ given by $\mu(w_t) = w_{t+1}$. Since B_k is obtained from B_m by the rotation, defined by μ^{k-m} then H_k is isomorphic to H_m. By Theorem 3, H_m contains a Hamilton path I_m with ends T_m and T'_m; therefore H_k contains a Hamilton path I_k with ends $\mu^{k-m}(T_m)$ and $\mu^{k-m}(T'_m)$.

Since for $k = 1, 2, \ldots, m$, all paths $\mu^{k-m}(T_m)$ and $\mu^{k-m}(T'_m)$ are obtained from the cycle $C = w_1, w_2, \ldots, w_{2m}, w_1$ by deleting an edge, then they are pairwise adjacent in G_{2m}. Therefore $I_1, I_2, \ldots, I_m$ can be connected to obtain a Hamilton cycle of G_{2m}.

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