Computational Investigations of the Optimality of Two- and Three-Dimensional Triangulations under Several Criteria

(Extended Abstract)

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Abstract

The optimality of triangulations in two and three dimensions is computationally investigated by using integer programming (IP).

Two formulations of triangulations are introduced, one based on the independent set problem, and the other based on the set partitioning problem. Some properties of triangulations that are interesting from a theoretical or practical point of view are considered as objective functions for IP. Finally, some computational results are given.

Introduction

It is well known that Delaunay triangulation (DT) in two dimensions is optimal in many respects; for example, it maximizes the minimum angle, and minimizes the largest circumsphere and the largest minimum enclosing sphere. Many applications, such as meshing algorithms for the finite element method (FEM), depend on this characteristic. But in three dimensions, it only minimizes the minimum enclosing sphere [1], and does not necessarily lead to optimality in other respects. In other words, DT is not so good in three or more dimensions. Thus, it is necessary to obtain a good triangulation for each application, namely, to optimize the triangulation.

In this paper we investigate the optimality of triangulations in two and three dimensions by using integer programming formulations. We first introduce two formulations. One is based on the independent set problem, which was introduced in [2] as a means of obtaining the minimum weight triangulation in two dimensions. We extend it to higher dimensions. The other is based on the set partitioning problem (SPP). The convex hull of a point set is divided into small cells (called chambers) separated by the facets of all the candidate triangles, and obtaining the triangulation can be treated as an instance of the SPP by regarding each chamber as an element [3]. De Loera et al. [4] proved that the cocircuit form equations given in the following section are sufficient to describe the SPP based on chambers.

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We then consider several objective functions in both two and three dimensions, which are interesting from a theoretical or practical point of view. We also give some computational results.

Formulations

The first formulation is based on the independent set problem. For example, in two dimensions, two intersecting edges e_i, e_j cannot appear in a triangulation together, and therefore we can set constraints on the corresponding variables as $y_i + y_j \leq 1$. When triangle t_j appears in a triangulation, each of its facets (edges) e_i also appears, and thus $y_i - x_j \geq 0$ (x_i corresponds to triangle t_i). M is the number of triangles in a triangulation and is constant in two dimensions:

$$\textbf{minimize} \sum_i c_i x_i$$
$$\begin{array}{rrcl} \text{s.t.} & x_i, y_i & \in & \{0,1\} \\ & y_i + y_j & \leq & 1 \text{ (edges } e_i \text{ and } e_j \text{ intersect)} \\ & y_i - x_j & \geq & 0 \text{ (edge } e_i \text{ is a facet of triangle } t_j) \\ & \sum_i x_i & = & M \end{array}$$

This is a formulation in two dimensions; in higher dimensions, the intersections are defined among simplices in different dimensions, and the facet relations should be defined for every simplex. Further, the constraint on the number of triangles is replaced by the total volume constraint, since the cardinality varies in dimensions higher than two.

The second formulation is based on the SPP, and can be written as follows[4] : Let e_k be a $(d-1)$-simplex. Two half spaces $\mathcal{H}^+_{e_k}, \mathcal{H}^-_{e_k}$ are defined by a hyperplane $\mathcal{H}_{e_k}$ containing e_k. Let $e_k \cup a$ be a d-simplex t_i defined by an element a of the point set $\mathcal{A}$ and e_k.

$$\textbf{minimize} \sum_i c_i x_i$$
$$\text{s.t.} \quad x_i \in \{0,1\}$$
$$\sum_{t_i = e_k \cup a, a \in \mathcal{A} \cap \mathcal{H}^+_{e_k}} x_i - \sum_{t_i = e_k \cup a, a \in \mathcal{A} \cap \mathcal{H}^-_{e_k}} x_j$$
$$= \begin{cases} \pm 1 & e_k \text{ is on the convex hull of } \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$$

Objective Functions

In applications such as FEM, we should avoid flatness of triangles for the sake of computational stability and accuracy. The most straightforward index of the flatness is the aspect ratio (AR), which is the ratio of the radii of the circumscribing and inscribing spheres. Thus, we minimize **the maximum AR (mAR)**; further, since the number of triangles is constant in two dimensions, we minimize **the sum of all the ARs (sAR)**.

The number of triangles (C) varies in three dimensions, and is an interesting objective function related to the following open problem:

Open Problem 1 *[5] Is there a polynomial-time algorithm for triangulating an arbitrary convex polyhedron with the minimum number of tetrahedra?*

The minimum weight triangulation in two dimensions is a famous problem, since it is not known whether it is solvable in polynomial time [6]. In three dimensions, **the sum of the weights (W)** is extended to the sum of the surface areas of tetrahedra.

Table 1: Optimality of triangulations in 2D

Input		DT	IP(mAR)	IP(sAR)	IP(W)
20	mAR	524.94	**524.94**	524.94	524.94
	sAR	739.99	1633.54	**730.44**	804.30
	W	19.246	24.961	20.548	**18.994**
40	mAR	22632.5	**22632.5**	22632.5	22632.5
	sAR	23465.9	79428.2	**23378.3**	23875.1
	W	39.050	62.680	40.511	**38.148**
60	mAR	749179.7	**749179.7**	749179.7	749179.7
	sAR	774087.4	2021213.6	**774070.2**	779885.5
	W	50.612	82.064	51.633	**49.720**

Table 2: Optimality of triangulations in 3D

Input		DT	IP(mAR)	IP(C)	IP(W)
10	mAR	3112.8	**79.8**	1617.7	452.7
	C	19	15	**9**	12
	W	7.894	7.630	6.833	**6.503**
20	mAR	29346.6	**488.2**	1617.7	3070.3
	C	66	56	**33**	43
	W	17.563	18.960	17.500	**13.875**
30	mAR	2637.7	—	97120.7	1.238×10^8
	C	115	—	**58**	76
	W	25.400	—	29.112	**19.765**

Experiments and Remarks

Tables 1 and 2 show the optimality of triangulations in two and three dimensions. Bold digits denote optimal values. The input points were randomly generated in a unit square and a unit cube.

We could not solve the problem of minimizing the mAR for 30 points in three dimensions because it consists of a large-scale bottleneck optimization: the branch and bound procedure just went deeper and the lower bound never improved. It may be possible to solve this problem by using binary search. In two dimensions, all the triangulations had the same mAR, because the value is highly dependent on a point very close to but not on an edge of the convex hull. However, there are cases in which the mAR is variable. DT maximizes the minimum angle, and hence minimizes the mAR in most cases, but not always; in fact we were able to generate a counterexample.

Optimization based on other criteria may be classified as summation-style, as opposed to bottleneck optimization. For the former, the integer programming approach is suitable, whereas for the latter, the algorithmic approach is suitable. In two dimensions, DT gave values close to the optimal under summation-style criteria. On the other hand, in three dimensions, a DT tends to contain more tetrahedra and the weight is not small. DT can contain a very flat tetrahedron, compared with other triangulations, but this seems to depend on the point set, and more experiments would be necessary to confirm this tendency. Anyway, we can observe that DT is not close to optimality in three dimensions.

Another very important property of triangulation in three dimensions is that not all polyhedra can be triangulated [7]. Through the use of integer programming and maximization of the volume that can be triangulated, it may be possible to gain some insight into triangulatability.

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