Hyperplane Depth and Nested Simplices (Extended Abstract)

William Steiger
Computer Science
Rutgers University

Rephael Wenger*
Computer and Information Science
The Ohio State University

Abstract

Given a set of n hyperplanes in R^d and a unit vector $u \in R^d$, the directional-depth of the point $p \in R^d$, in direction u, is the number of hyperplanes met by the ray from p in direction u (possibly at infinity.) The *hyperplane depth* is the smallest directional depth, the minimum taken over all directions. We prove that there is always a point with depth $c_d n$, where $c_d > 0$ is a constant independent of n. We prove this via a stronger statement for hyperplanes in general position, namely that such a set of hyperplanes may be partitioned into $a_d n$ sets of size $d+1$ each so that the $a_d n$ simplices formed by this partition have nonempty intersection.

1 Introduction and Summary

Let $\mathcal{H}$ be a set of n hyperplanes in R^d and let $u \in R^d$ be a unit vector. The *directional depth* $d_u(p, \mathcal{H})$ of a point $p \in R^d$, in direction u, is the number of the hyperplanes that are met by the ray from p in direction u. If u is orthogonal to the normal vector of $h \in \mathcal{H}$, a ray in direction u meets h at infinity. The *hyperplane depth*

$$HD(p, \mathcal{H}) = \min_u d_u(p, \mathcal{H})$$

is the smallest directional depth. This notion was developed by Rousseeuw and Hubert [3] as a measure of quality of regression fits in statistics: the hyperplane $h \in R^d$ that has been computed as a fit to n given points $p_1, \ldots, p_n$ in R^d is evaluated by its *regression depth* which is simply its depth HD (as a point in the arrangement of hyperplanes dual to $p_1, \ldots, p_n$). For simplicity we sometimes use $d_u(p)$ and $HD(p)$ in place of $d_u(p, \mathcal{H})$ and $HD(p, \mathcal{H})$, respectively.

*Supported in part by NSA grant MDA904-97-1-0018 and by DIMACS.

For $d = 1$ the median is a point of maximal "hyperplane" depth $\lceil n/2 \rceil$. Little is known about hyperplane depth in dimension $d > 2$. However in the plane we have

Proposition 1 (Rousseeuw and Hubert [3]) *Given n lines in the plane there is a point p with*

$$HD(p) \geq \lceil \frac{n}{3} \rceil$$

and the bound is sharp.

In the same paper the natural generalization was conjectured, namely that hyperplane arrangements in R^d always have points of depth at least $\lceil n/(d+1) \rceil$. Rousseeuw and Hubert also noted that the conjectured bound cannot be increased. Given n hyperplanes dual to n distinct points on the moment curve, $\{ (t, t^2, t^3, \ldots, t^d) : t \in R \}$, then, for all $x \in R^d$,

$$HD(p) \leq \lceil \frac{n}{d+1} \rceil.$$

In this paper we prove that the hyperplane depth of any arrangement of hyperplanes is at least $c_d n$ for some constant c_d dependent on d. In fact, for hyperplanes in general position, we prove a stronger theorem about a point lying in $a_d n$ simplices formed from the arrangement. An arrangement of $n \geq d$ hyperplanes is in general position in R^d if every set of d hyperplanes meet in a single point in R^d, but *no* $d+1$ hyperplanes have a common point of intersection. A set of $d+1$ hyperplanes in general position in R^d determine a simplex bounded by the $d+1$ hyperplanes.

Theorem 1 *For each d there is a constant $a_d > 0$ so that any n hyperplanes in general position in R^d may be partitioned into $a_d n$ sets, $d+1$ hyperplanes in each set, in such a way that that the $a_d n$ simplices they determine have nonempty intersection.*

Our proof of Theorem 1 uses the Zone Theorem for hyperplane arrangements. A set $\mathcal{H}$ of hyperplanes in R^d, partitions R^d into open sets of dimension d, called *cells*, and relatively open lower dimensional faces. The *zone* of a hyperplane g (not necessarily in $\mathcal{H}$) is the set of all cells which intersect g. The *complexity of a cell* is the number of faces it has. The *complexity of a zone* is the sum of the complexity of all the cells in a zone. The *Zone Theorem* states that the complexity of a zone in an arrangement of n hyperplanes in R^d is $O(n^{d-1})$.

We actually only need a bound on the number of vertices in a zone. For a cell C, let $v(C)$ be the number of vertices of C. Let $v(g;\mathcal{H})$ be the sum of $v(C)$ over all cells C in the zone of hyperplane g. Note that a vertex may appear in more than one cell in a zone of C and may be counted more than once in $v(g;\mathcal{H})$. Let $v(n,d)$ be the maximum of $v(g;\mathcal{H})$ over all hyperplanes g and sets $\mathcal{H}$ of n hyperplanes in R^d.

Theorem 2 (Edelsbrunner, Seidel, Sharir [1]) *For all $n, d > 0$,*

$$v(n,d) \leq \frac{2}{d} b_d \binom{n}{d-1}$$

where $b_d = \frac{2}{3}6^{d-1} + \frac{3}{4}2^{d-1}$.

The following lemma follows directly from the Theorem 2.

Lemma 1 *An arrangement of $n > 4b_d + d - 1$ hyperplanes in general position in R^d has some vertex which lies in the interior of a simplex determined by some $d+1$ hyperplanes of the arrangement.*

Proof: Let $\mathcal{H}$ be a set of $n > 4b_d + d - 1$ hyperplanes in general position in R^d. Let S be the set of vertices of the arrangement of $\mathcal{H}$ and let g and g' be two parallel hyperplanes containing all the vertices of S between them and in general position with respect to all the hyperplanes in $\mathcal{H}$. Every unbounded cell intersects either g or g', or both. By Theorem 2, the total number of vertices in the zones of g and g' is at most $2(\frac{2}{d}b_d\binom{n}{d-1})$. The cells in the zones of g and g' are the unbounded cells in the arrangement of $\mathcal{H}$. The arrangement of $\mathcal{H}$ contains $\binom{n}{d}$ vertices. For $n > 4b_d + d - 1$,

$$2(\frac{2}{d} b_d \binom{n}{d-1}) < \binom{n}{d}.$$

Thus there must be some vertex w which does not lie in an unbounded cell. Let $\mathcal{H}'$ be the d hyperplanes whose intersection is vertex w. Vertex w must lie in a bounded cell C in the arrangement of $\mathcal{H} \setminus \mathcal{H}'$. By a simple application of either Helly or Carathéodory's Theorem, there are $d+1$ hyperplanes in $\mathcal{H} \setminus \mathcal{H}'$ which define a simplex containing C. Thus w lies in the interior of a simplex determined by $d+1$ hyperplanes of H. □

Proof of Theorem 1 Let $\mathcal{H}$ be a set of n hyperplanes in general position in R^d. Let $\gamma_d = \lfloor 4b_d \rfloor + d$. By Lemma 1, any subarrangement of γ_d hyperplanes of $\mathcal{H}$ contain a vertex lying in the interior of a simplex determined by the subarrangement. There are $\binom{n}{\gamma_d}$ such subarrangements. Each vertex, simplex pair may lie in $\binom{n}{\gamma_d - 2d - 1}$ different subarrangements, so the number of such pairs is at least

$$\frac{\binom{n}{\gamma_d}}{\binom{n}{\gamma_d - 2d - 1}}.$$

Since there are $\binom{n}{d}$ vertices, the average number of simplices per vertex is at least

$$\frac{\binom{n}{\gamma_d}}{\binom{n}{\gamma_d - 2d - 1}\binom{n}{d}}.$$

Some vertex w is contained in the interior of that number of simplices. Each hyperplane may be used to define $\binom{n}{d}$ simplices and there are $d+1$ bounding hyperplanes per simplex. From this set of simplices containing w, choose a maximal set such that no two simplices share a defining hyperplane. This set has size at least

$$\frac{\binom{n}{\gamma_d}}{\binom{n}{\gamma_d - 2d - 1}\binom{n}{d}(d+1)\binom{n}{d}} \geq a_d n,$$

for some $a_d > 0$. Thus $\mathcal{H}$ can be partitioned into $a_d n$ sets, each containing $d+1$ hyperplanes, such that the simplex defined by each set contains w. □

Theorem 3 *For each d there is a constant $c_d > 0$ so that for every n hyperplanes in R^d there is a point p with*

$$HD(p) \geq c_d n.$$

For hyperplanes in general position, the conclusion is true by Theorem 1. The proof for the general case appears in the full version.

References

[1] H. Edelsbrunner, R. Seidel, and M. Sharir. On the Zone Theorem for Hyperplane Arrangements *SIAM J. Computing 22*, 418-429. (1993),

[2] J. Gill, W. Steiger, and A. Wigderson. Geometric Medians, *Discrete Math. 108*, 37-51. (1992),

[3] P. Rousseeuw and M. Hubert. *Depth in an Arangement of Hyperplanes*, manuscript, 1998.