A Modified Longest Side Bisection Triangulation

Christoph Stamm Stephan Eidenbenz Renato Pajarola

Computer Science Department
Institute of Theoretical Computer Science
ETH Zurich, Switzerland
E-mail: stamm@inf.ethz.ch

Abstract

Fast and efficient exploration of large terrains in a virtual reality manner requires different levels of detail to speedup processing of terrain parts in background. Thus, the triangulation refinement problem has become an important issue in processing terrain data.

The longest side bisection of triangles is one possible way to subdivide a triangle in a couple of subtriangles. In the larger context of triangulations, this subdivision can be seen as an approach of the triangulation refinement problem. This particular triangulation refinement yields a strong hierarchical triangulation, which itself is an approach of a multiresolution model for terrain surfaces.

In its pure form, the longest side bisection of a triangle t (assume α is a smallest angle in t) and its descendants only produces triangles whose smallest angles are always greater or equal to $\frac{\alpha}{2}$.

In this paper we show that with a small modification, the lower bound on the smallest angles can be increased to $\frac{2}{3}\alpha$, without increasing the total number of subtriangles produced in two consecutive refinement steps. Even the new vertices inserted in the triangulation are exactly the same.

1. Introduction

Triangulated irregular networks (TINs) [6] are widely used to represent terrain surfaces, e.g. in geographical information systems (GIS). Vertices in TINs describe nodal terrain features, e.g. peaks, pits or passes, while edges depict linear terrain features, e.g. break, ridge or channel lines [6].

Hierarchical triangulation [2] (HT, introduced by Floriani et al. in 1984) is a method to describe a terrain in a number of different resolutions. The concepts of so called multiresolution models and detailed algorithms to produce a multiresolution model can also be found in [1]. Using the method of hierarchical triangulation, the terrain is approximated in successively finer levels of detail by triangular patches whose projections in the horizontal plane are nested. If we restrict ourselves such that one triangle is exactly partitioned into a set of triangles in each step, then we call the HT strict (SHT), and we get a tree structure. The HT method guarantees that the difference in elevation of any location between terrains described by the resulting TIN and the original grid digital elevation model (DEM) will never exceed a prespecified precision level.

The triangulation problem is the following [9]: Given N representative points of a polygonal region, join them by non intersecting straight line segments, such that every region internal to the polygon is a triangle.

Many criteria have been proposed as to what constitutes a 'good' triangulation. The criteria are either of topographical or geometrical characteristic. Typical geometrical characteristics are: maximizing the smallest angle or minimizing the total edge length. The Delaunay triangulation [3] satisfies the property of maximizing the smallest angle, but there is no refinement process known, leading to a strict hierarchical Delaunay triangulation. In our applications (visualizing large terrains [8]) we are interested in triangles without small angles, because small angles have some drawbacks in visualization, e.g. long, disturbing edges, flickering, and shading artifacts.

The triangulation refinement problem can be formulated as follows [9]: Given a non-degenerate triangulation, construct a locally refined triangulation with a desired resolution or a desired maximum error such that the smallest angle is bounded.

Analogous to the triangulation problem there are several criteria for a 'good' refinement strategy. For instance, minimizing the number of refinement steps to reach the desired resolution/maximum error, minimizing the number of subtriangles in one refinement step (to gain a maximum adaptivity), or leading to a SHT. In our applications we are interested in strict hierarchical triangulations (SHT) with maximum adaptivity in terms of the number of triangles.

This paper is organized as follows: In Section 2 we describe a refinement method, called longest side bisection of triangles, leading to a SHT with triangles of 'guaranteed quality'. The nice properties of longest side bisection are welcome in the real-time visualization of large TINs. In this real-time visualization setting an initial Delaunay triangulation combined with the longest side bisection refinement method [10] seems to be a very appropriate multiresolution model approach. In Section 3 we describe our modification of longest side bisection of triangles and we prove a new better lower bound on the angles of triangles constructed by our method. In Section 4 we discuss progressive mesh refinement based on our modified longest side bisection method. And last, in Section 5, we conclude our results.

2. Longest side bisection of a triangle

Definition 2.1 *The longest side bisection of a triangle t is the partition of t by the straight line segment from the midpoint of its longest edge to the opposite vertex.*

Definition 2.2 *The neighbor of t is the neighboring triangle $\bar{t}$ which shares with t a longest side (the candidate for bisection) of t.*

Definitions 2.3 *Two triangulations τ_i, τ_j are said to be adjacent along a straight line segment l if their domains intersect only along l; τ_i and τ_j are said to be matching along l if they are adjacent along l and if each vertex on l is both a vertex of τ_i and a vertex of τ_j; a triangulation τ is said to be matching if all adjacent triangulations $\tau_i, \tau_j \in \tau$ are matching.*

Definition 2.4 *The longest side bisection triangulation is a strict hierarchical matching triangulation, where each refined triangle is subdivided using the method of longest side bisection.*

A triangle subdivision based on the longest side bisection solves the triangulation refinement problem with maximum adaptivity, because in each refinement step only two new subtriangles are constructed. It also leads to a SHT, since every single triangle can be subdivided into a pair of smooth TINs whose geometrical properties only depend on the initial triangulation [9]. Furthermore, since the point location in a triangulation of size N takes $O(\log N)$ time and the work for one point insertion uses only constant time, the insertion of k points can be performed in time $O(k \log N)$.

However, this recursive mesh refinement can produce non-matching triangulations. Thus in order to make the triangulations matching, the local subdivision of a given triangle t involves a refinement of its neighbor $\bar{t}$. We bisect t and its neighbor $\bar{t}$ and continue this process iteratively until the last two triangles share the same longest side. The same idea has to be applied in order to match the set of non-matching vertices generated in the inverse order in which they were created. This triangulation refinement process is sometimes called Rivara refinement [9].

If the initial polygonal region is a square, which is splitted in two rectangular, isosceles triangles, then the described refinement method leads to a restricted quadtree triangulation (RQT) structure which is thoroughly described in [7, 12]. The longest side bisection triangulation belongs to the larger class of *bintree triangulations* and is a generalization of the triangulation presented in [4] (that is identical to the RQT).

3. Modified longest side bisection

Let ΔABC (see Fig. 1) be a given triangle with interior angles α, β and γ located at A, B and C, respectively. If ΔABC is bisected into two triangles $\Delta A_i B_i C_i$ with interior angles α_i, β_i and γ_i, $i = 1, 2$, we use both the notations

$$(\alpha, \beta, \gamma) \longrightarrow (\alpha_i, \beta_i, \gamma_i) \text{ and } (\alpha_i, \beta_i, \gamma_i) \longleftarrow (\alpha, \beta, \gamma).$$

(α, β, γ) denotes a similiarity class of triangles with interior angles α, β and γ and '$\longrightarrow$' is a binary relation on the set of all these similarity classes. We also use the notation $\overline{MN}$ to denote the line segment between the points M and N and $|\overline{MN}|$ to denote its Euclidean distance.

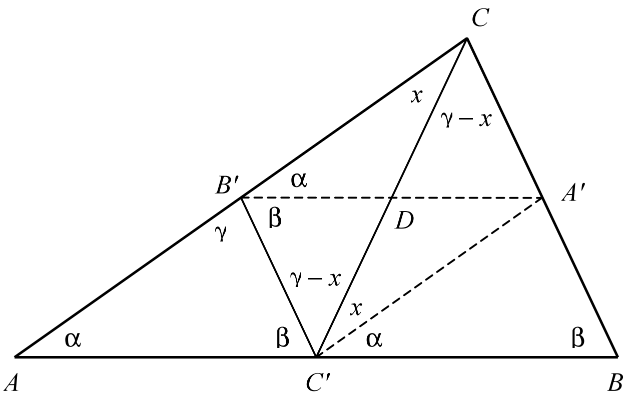

Figure 1. Bisections of a triangle ΔABC

Let A', B' and C' be the midpoints of $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$, respectively. Without loss of generality we assume that $0 < \alpha \leq \beta \leq \gamma$. Since the sizes of the edges of ΔABC are in the same relation as the opposite angles, it follows directly $|\overline{BC}| \leq |\overline{AC}| \leq |\overline{AB}|$. $|\overline{AC}| \geq |\overline{BC}|$ yields $|\overline{AC}| > |\overline{CC'}|$. It follows in combination with $|\overline{AC}| > |\overline{AC'}|$ that $\overline{AC}$ is the longest side in $\Delta AC'C$. From Fig. 1, we obtain:

$$
\begin{array}{ccc}
(\alpha,\beta,\gamma) & \longrightarrow & (\alpha+x,\beta,\gamma-x) \\
\downarrow\uparrow & & \\
(\alpha,\beta+\gamma-x,x) & \longrightarrow & (\alpha+\beta,\gamma-x,x)
\end{array} \tag{1}
$$

These relations are valid in general.

Lemma 3.1 *If*

$$|\overline{BC}| \geq |\overline{CC'}| \geq |\overline{BC'}|, \tag{2}$$

then all three edges of triangle ΔABC would be bisected in two consecutive refinement steps.

Proof $\overline{AB}$ is the longest side of triangle ΔABC. Thus in the first step ΔABC is bisected into two triangles $\Delta AC'C$ and $\Delta BCC'$ (see Fig. 1). We also know that $\overline{AC}$ is the longest side of $\Delta AC'C$ and is therefore bisected in the second refinement step. Because of (2) $\overline{BC}$ is a longest side of $\Delta BCC'$, and therefore, it will be bisected in the second refinement step too. □

Definition 3.2 *The modified longest side bisection of a triangle ΔABC is equal to the longest side bisection of ΔABC, except when (2) holds. In case where (2) holds, ΔABC is subdivided into four equal triangles (they are all similar to ΔABC) using line segment $\overline{A'B'}$ instead of $\overline{CC'}$. This particular subdivision is often called 'quaternary triangulation'.*

Definition 3.3 *$T(A,B,C)$ is an infinite family of triangles, formed by iteratively applying the modified longest side bisection to triangle ΔABC and to its descendants.*

Theorem 3.4 *Let α be the smallest interior angle of triangle ΔABC. If Δ is a triangle in $T(A,B,C)$, and θ is an interior angle of Δ, then $\theta \geq \frac{2}{3}\alpha$.*

The proof consists of two parts. In the first part we prove that only one refinement step applying the modified longest side bisection does not falsify the theorem. In the second part we show that the theorem is also true if we iteratively apply the modified longest side bisection to triangle ΔABC and its descendants.

Proof (Part 1) If (2) is valid (see region ① in Fig. 2), then the precondition of the modified longest side bisection holds, and the smallest interior angle $\forall\Delta \in T(A,B,C)$ is by definition equal to α, because of the quaternary triangulation applied in this case.

Otherwise, if (2) is not satisfied, we have to check two different cases: $\gamma \geq \frac{\pi}{2}$ (see region ② in Fig. 2) and $\gamma < \frac{\pi}{2}$

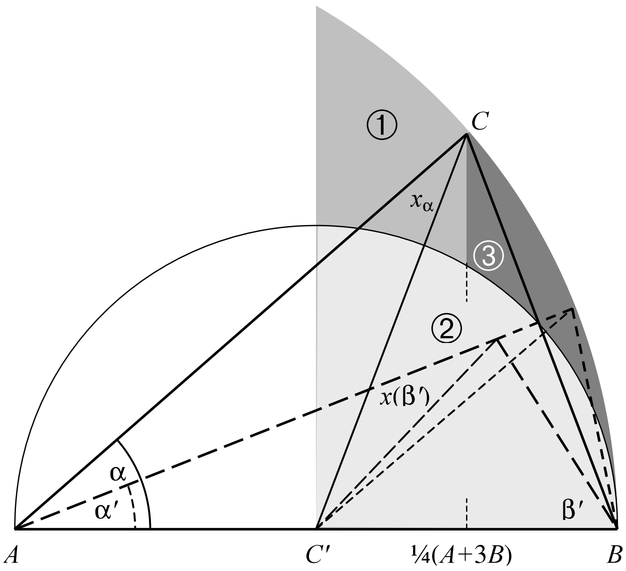

Figure 2. Various values of the angle x

(see region ③ in Fig. 2). But before we can do this, we need some intermediate results.

We assumed without loss of generality that $0 < \alpha \leq \beta \leq \gamma$ and since $\alpha+\beta+\gamma = \pi$, it follows that

$$\alpha \leq \frac{\pi}{3} \leq \gamma < \pi \quad \text{and} \quad \beta < \frac{\pi}{2}. \tag{3}$$

The relations $|\overline{B'C}| = \frac{1}{2}|\overline{AC}| \geq \frac{1}{2}|\overline{BC}| = |\overline{A'C}|$ yield

$$x \leq \gamma - x. \tag{4}$$

Since also $\beta \geq \alpha$ and $|\overline{AC}| + |\overline{BC}| > |\overline{AB}|$, it follows $|\overline{AC}| > |\overline{AC'}|$ and

$$\gamma - x + \beta \geq \frac{\pi}{2}. \tag{5}$$

In Fig. 1, $\gamma - x + \beta \geq \frac{\pi}{2} > \alpha$, $x + \alpha > \alpha$, and $\pi - \gamma = \alpha + \beta > \alpha$. Thus, the only candidates for angles less than α are x and $\gamma - x$.

1. $\gamma \geq \frac{\pi}{2}$: If $\gamma \geq \frac{\pi}{2}$, then $|\overline{AC'}| \geq |\overline{CC'}|$, therefore

$$x \geq \alpha \Leftrightarrow \gamma \geq \frac{\pi}{2}. \tag{6}$$

It follows from (4) and (6) that $\gamma - x \geq x \geq \alpha$, and hence all interior angles of $\Delta AC'C$ and $\Delta BCC'$ are greater or equal to α. Thus, the modified longest side bisection of ΔABC does not decrease the smallest interior angle in the two subtriangles.

2. $\gamma < \frac{\pi}{2}$: If $\gamma < \frac{\pi}{2}$, then $|\overline{AC'}| < |\overline{CC'}|$, and since (2) is not satisfied, it holds that

$$|\overline{BC}| < |\overline{CC'}|, \tag{7}$$

and hence $\alpha + x < \beta \leq \gamma$. It follows that

$$\gamma - x > \alpha. \tag{8}$$

Therefore, the only remaining angle that can be smaller than α is x. Since we are interested in a lower bound for $\frac{x}{\alpha}$, we first have to find the minimum of x for a fixed α. Let x_α be this minimum. With reference to Fig. 2, let us fix A, B, and α, and change β between $\beta_{min} = \alpha$ and $\beta_{max} = \gamma$. Clearly, $x = x(\beta)$ is a decreasing function of β in the region $\beta_{min} \leq \beta \leq \beta_{max}$. Thus, with $\beta + \gamma = \pi - \alpha$ and $\beta \leq \gamma$, x is minimal for a fixed α if $\beta = \gamma = \frac{\pi - \alpha}{2}$. With a little trigonometry [11], we obtain

$$\tan x_\alpha = \frac{\sin \alpha}{2 - \cos \alpha}. \tag{9}$$

Note that x_α is an increasing function of α in the region $0 < \alpha \leq \frac{\pi}{3}$ (see Fig. 2), which can be easily verified by computing the derivative of x_α, using (9). Because vertex C lies in region ③, we are only interested in α in the region $0 < \cos \alpha \leq \frac{3}{4}|\overline{AB}|/|\overline{AC}|$. If α is small, x_α is a better lower bound than $\frac{2}{3}\alpha$, since $\lim_{\alpha \to 0} \frac{x_\alpha}{\alpha} = 1$. Finally, we get our lower bound for $\frac{x_\alpha}{\alpha}$ if we set $\beta = \gamma$ and hence $|\overline{AC}| = |\overline{AB}|$ and $\cos \alpha = \frac{3}{4}|\overline{AB}|/|\overline{AC}| = \frac{3}{4}$:

$$\tan x_\alpha = \frac{\sqrt{1 - \cos^2 \alpha}}{2 - \cos \alpha} = \frac{\sqrt{7}}{5} \tag{10}$$

$$\frac{x_\alpha}{\alpha} = \frac{\arctan \frac{\sqrt{7}}{5}}{\arccos \frac{3}{4}} \simeq 0.67 > \frac{2}{3} \tag{11}$$

□

Now, for one refinement step we have shown that x is larger than $\frac{2}{3}\alpha$, and that all interior angles of triangle $\Delta BCC'$ are greater or equal to α. Next, we have to prove that the iterative refinement of triangle ΔABC does not yield an angle smaller than x.

Proof (Part 2) Because one refinement step of ΔABC does not yield an angle smaller than x, we know that the further refinement of $\Delta BCC'$ does not yield an angle smaller than x too, because all angles in $\Delta BCC'$ are larger than α. To show that the iterative refinement of ΔABC does not falsify the theorem, we have to prove that the further refinement of a triangle with an interior angle x does not yield an angle smaller than x. Therefore, we focus on triangle $\Delta AC'C$ with point C lying in region ③.

We already know that $\overline{AC}$ is the longest side in $\Delta AC'C$ and is therefore bisected in the second refinement step of this triangle. Since we are only interested in subtriangles with at least one angle smaller than α, we omit $\Delta AC'B'$, because $\Delta AC'B'$ is similar to ΔABC (see Fig. 1). To prove that a further refinement of subtriangle $\Delta B'C'C$ does not yield an angle smaller than x, we bisect $\Delta B'C'C$ once more. The relations $|\overline{CC'}| > |\overline{AC'}| = \frac{1}{2}|\overline{AB}| \geq \frac{1}{2}|\overline{AC}| = |\overline{CB'}|$ yield

$$\alpha + \beta > \gamma - x. \tag{12}$$

Because of (4) and (12), $\overline{CC'}$ is the longest side of $\Delta B'C'C$, and is therefore bisected. Let D be the midpoint of $\overline{CC'}$. With reference to Fig. 1, $\Delta DB'C'$ is similar to $\Delta C'BC$ and $\Delta B'DC$ is similar to $\Delta AC'C$. Thus, we obtain:

$$\begin{array}{ccc} (\alpha, \beta, \gamma) & \longrightarrow & (\alpha + x, \beta, \gamma - x) \\ \downarrow\uparrow & & \uparrow \\ (\alpha, \beta + \gamma - x, x) & \rightleftarrows & (\alpha + \beta, \gamma - x, x) \end{array} \tag{13}$$

The configuration in (13) is such that arrows going outside of it can originate only at $(\alpha + x, \beta, \gamma - x)$. Because of (8), all angles of $(\alpha + x, \beta, \gamma - x)$ are larger than α. □

Let us now summarize the proof of the theorem. In each refinement step of triangle ΔABC, we only get interior angles greater or equal to α, except when point C belongs to region ③ . In this case, we have shown in (11) a new lower bound for x of $\frac{2}{3}\alpha$. Additionally, we have shown in (13) that further refinements of a triangle with angle x does not yield triangles with a smaller angle than x. Therefore, also the recursive modified longest side bisection refinement $T(A, B, C)$ does not produce any angles smaller than $\frac{2}{3}\alpha$.

4. Progressive mesh refinement

The modified longest side bisection of a triangle as defined in the previous section has some drawbacks, because it sometimes splits the triangle in four instead of only two subtriangles. For that reason, its usage in mesh refinement is limited. To omit this disadvantage, we define the following refinement rule:

Definition 4.1 *The modified longest side bisection rule: Assume triangle t is in the first step redefined by applying the longest side bisection method. If (2) holds for t and both descendants of t have to be refined, then t is subdivided by applying the modified longest side bisection method.*

A continuous refinement of the triangle mesh τ can efficiently be achieved by an iterative subdivision of a longest edge of all triangles t in τ. For that purpose we maintain all edges of the current triangulation in a heap that has a longest edge at its root, and keeps the smaller ones further down in the heap. The refinement step picks a longest edge from the root of the edge-heap and performs the subdivision on the two incident triangles, taking into account the modified longest side bisection rule. Note that this longest edge of the current triangulation is indeed a longest edge of both incident triangles. The edge selection and the refinement step can be performed in constant time $O(1)$. However, the heap update costs $O(\log n)$ time, because the resulting two new edges from the refinement step have to be inserted into the heap.

A way of refining the triangle mesh more adaptively is to choose the edge that has to be split not because of its length, but based on the largest approximation error of all edges (i.e. distance to the surface that has to be approximated). This adaptive mesh refinement is already described in [9], however, we will briefly discuss the split propagation behavior.

Lemma 4.2 *If we split the common edge e of two adjacent triangles, then the propagated refinements only split edges that are longer than e.*

Proof If e is the longest edge in a triangle t, then no split-propagation occurs at all in t, since the triangle t is correctly refined according to the modified longest side bisection rule.

If e is the second longest side of t, then we also have to split the longest side of t because of the modified longest side bisection rule. However, the smallest side of t does not have to be split at all, see also Fig. 1. Let $e = \overline{AC}$ be the second longest side of $t = \Delta ABC$. Because of (1) we get a subdivision into three triangles, where the smallest side of t, $\overline{BC}$, is not bisected.

If e is the smallest edge then the split will obviously only propagate to longer edges of t.

Therefore, the split propagates always to an edge that is longer than e. The same is obviously true if e is the smallest side of t. □

Therefore, a local mesh refinement can cause splits that propagate to growing edges, and thus to larger triangles or larger angles. As soon as the split propagation arrives at a longest side of a triangle, the propagation stops there. In contrast to the RQT we cannot estimate the split propagation in general, however, it is somehow determined by the initial angles of the subdivided triangles.

Note that both discussed mesh refinement methods can locally lead to $\frac{\alpha}{2}$ smallest angles in the first step of the modified longest side bisection rule. Only if both descendants of the first refinement step are further refined, applying the second step of the modified bisection rule, then we get at least $\frac{2}{3}\alpha$ angles.

Progressive meshing can efficiently be achieved by a sequence of refinement events. In contrast to [5], these update events are edge bisections in our case. Each update of splitting two adjacent triangles up into four can be performed in $O(1)$ time, and affects the triangulation only locally. Furthermore, mesh morphing can easily be incorporated: The new vertex v' of a bisected edge $e = v_2 - v_1$ is linearly interpolated to $v' = \frac{1}{2}(v_1 + v_2)$, and then smoothly morphed to its final position v using a blending function $f(s)$ that is monotonically increasing for $s = [0.0, 1.0]$. Therefore, the current intermediate vertex position using the blending function is: $vcurrent = f(s)v + (1 - f(s))v'$.

For a given triangulation hierarchy the split propagation can be encoded by dependency relations similar to the RQT. Every edge subdivision that indeed is a longest side bisection of triangle t only depends on the opposite vertex of this edge in t. However, the subdivision of a smaller edge e in t depends on the longest side bisection of t. Therefore, each midpoint of an edge e has two dependencies pointing to the opposite vertex or to the midpoint of the longest side of the two adjacent triangles. This dependency relation can be computed during the construction of a longest side bisection triangulation hierarchy.

5. Conclusion

We have presented an adaptive hierarchical multiresolution triangulation based on the longest side bisection triangulation. This triangulation was introduced by [9, 11] and has the following nice properties:

- It only produces triangles whose smallest angles are always greater or equal to $\frac{\alpha}{2}$, where α is the smallest angle of the initial triangle.
- All produced triangles belong to a finite number of similarity classes of triangles.
- The Rivara refinement always terminates in a finite number of steps with the construction of a matching triangulation.

- It satisfies the following smoothness condition: for any pair of side-adjacent triangles $t_1, t_2 \in \tau$ (τ is a matching triangulation) with respective diameters h_1, h_2 it holds that $\frac{\min(h_1,h_2)}{\max(h_1,h_2)} \geq \delta > 0$, where δ only depends on the smallest angle of the initial triangulation.

Our modified longest side bisection refinement rule improves the lower bound of the smallest occurring angle to $\frac{2}{3}\alpha$. For interactive visualization of terrain surfaces it is important not to have small angles and thin triangles because of rendering artefacts as described in the introduction. Furthermore, we describe progressive meshing based on the longest side bisection triangulation, and examine the split propagation behavior.

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