The Size of the Open Sphere of Influence Graph in L_∞ Metric Spaces

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1 Introduction

Let V be a set of distinct points in some metric space. For each point $x \in V$, let r_x be the distance from x to its nearest neighbour, and let s_x be the open ball centered at x with radius equal to the distance from x to its nearest neighbour. We refer to these balls as the *spheres of influence* of the set V. The open sphere of influence graph $G = (V, E)$ is defined as the graph on V where (x, y) is an edge if and only if s_x and s_y intersect.

The maximum size of sphere of influence graphs (SIGs) in Euclidean space has been studied. Most notably, Avis and Horton proved that the number of edges of a SIG in the Euclidean plane is linear with respect to the number of vertices [1], and Guibas, Pach and Sharir extended the result to any fixed dimension [2]. However, little attention has been placed on SIGs in other metric spaces. In this paper, we focus on the size of open SIGs in d-dimensional metric spaces induced by the ∞-order Minkowski metric, where the spheres of influence are open hypercubes. We refer to these metric spaces as L_∞^d and present upper and lower bounds for the maximum number of edges in open L_∞^d-SIGs.

2 An upper bound on the number of edges

Lemma 1 *Let S be a collection of open balls in L_∞^d such that no ball in S contains the center of any other ball in S. Then for all points $p \in L_\infty^d$, p is contained in no more than 2^d balls of S.*

With the aid of Lemma 1, we would like to make our arguments by examining intersections which include the corners of the spheres of influence. However, since the spheres of influence are open and thus do not contain their corners, we instead choose to examine points which are inside the spheres and "close enough" to the corners. We refer to these points as ε-corners and define them as follows.

Let S be the spheres of influence of some point set, and let P be the set of all polytopes determined by pairwise intersections of balls in S. Let ε be one-half the smallest width over all polytopes in P. For each corner c of a ball in S, we define an ε-corner as $c + \varepsilon\hat{v}_{c\to o}$ where $\hat{v}_{c\to o}$ is the unit vector in the direction from the corner to the center of the ball.

Lemma 2 *Let X and Y be two intersecting spheres of influence in L_∞^d. Then the number of ε-corners of X plus the number of ε-corners of Y contained in $X \cap Y$ is at least two.*

Theorem 3 *No open sphere of influence graph of n vertices in L_∞^d has $(2^{2d-1}-2^{d-1})n$ edges or more.*

Proof *(sketch)***.** Let S be the spheres of influence of a point set V. Let C be the set of all points which are ε-corners of balls in S. Since S contains n balls, and each ball has 2^d ε-corners, C contains at most $2^d n$ points.

Let s_x be a ball in S such that p is a ε-corner of s_x. By Lemma 1, each point in C is contained in no more than 2^d balls. Since $p \in s_x$, p is contained in at most $2^d - 1$ balls of $S \setminus s_x$. Therefore, p cannot be in more than $2^d - 1$ pairwise intersections of balls involving s_x.

C contains $2^d n$ distinct ε-corners, each of which are involved in at most $2^d - 1$ intersections, and by Lemma 2, each intersection contains at least two ε-corners. Thus the sphere of influence graph on V contains at most

$$\begin{aligned} & 2^d n \times (2^d - 1) \div 2 \\ = \quad & (2^{2d} - 2^d)n \div 2 \\ = \quad & (2^{2d-1} - 2^{d-1})n \end{aligned}$$

edges. Since an extreme ε-corner of C cannot be involved in $2^d - 1$ intersections, we can be a bit more precise in stating that the graph must have fewer than $(2^{2d-1} - 2^{d-1})n$ edges.

□

3 A lower bound on the maximum number of edges

We construct a lattice in L_∞^d space such that each vertex has $(2^{2d+3} - 6d - 8)/9$ sphere of influence neighbours.

Let $\widehat{u_j}$ be the unit vector parallel to the j^{th} dimensional axis. We generate a lattice with the basis $\mathcal{B} = \{\vec{b_i} : 1 \le i \le d\}$ such that

$$\vec{b_i} = \widehat{u_i} + \delta_i \sum_{j<i} \widehat{u_j} - \delta_i \sum_{j>i} \widehat{u_j}$$

where each δ_i is a small positive real number, such that $0 < \delta_1 \le 1/4$, and $0 < \delta_j \le \frac{\delta_{j-1}}{4}$ for all $2 \le j \le d$.

Dimension (d)	Upper Bound	T_d	Gap btwn bounds
1	n	n	tight
2	$6n$	$6n$	tight
3	$28n$	$27n$	n
4	$120n$	$112n$	$8n$
5	$496n$	$453n$	$43n$
6	$2016n$	$1818n$	$198n$
$\lim_{d\to\infty}$			(1/9)Upper bound

Table 1: Bounds on the total number of edges in a L_∞^d-SIG.

Let this d-dimensional lattice be known as T_d.

Theorem 4 *In the infinite lattice T_d, each vertex has $(2^{2d+3}-6d-8)/9$ sphere of influence neighbours.*

Thus for finite SIGs of n vertices which are subsets of T_d, we achieve an asymptotic bound of $((2^{2d+2} - 3d - 4)/9)n$ edges. This leads us to the results in Table 1.

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References

[1] David Avis and Joe Horton. Remarks on the sphere-of-influence graph. *Discrete Geometry and Convexity*, 440:323–327, 1985.

[2] Leonidas Guibas, János Pach, and Micha Sharir. Sphere-of-influence graphs in higher dimensions. *Colloquia Mathematica Societatis János Bolyai*, 63:131–137, 1994.