Riemannian Computational Geometry — Voronoi Diagram and Delaunay-type Triangulation in Dually Flat Space —

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Abstract

One of most famous theorems in computational geometry is the duality between Voronoi diagram and Delaunay triangulation in Euclidean space. This paper proposes an extension of that theorem to the Voronoi diagram and Delaunay-type triangulation in *dually flat space*. In that space, the Voronoi diagram and the triangulation can be computed efficiently by using potential functions. We also propose higher-order Voronoi diagrams and prove that Delaunay-type triangulation is as good in dually flat space as it is in Euclidean space.

1 Introduction

When a computational geometry is extended to Riemannian space, geometric structures like those in Euclidean space are not always constructed, in some Riemannian spaces, there exist geometric objects that can be computed efficiently. In the space we deal with in this paper, called *dually flat space*, the duality between the *Voronoi diagram* and *Delaunay-type triangulation* can be proved. This duality is based on potential functions, hyperplane arrangement, and convex hull. In addition, we propose algorithms for computing these structures for given points by using potential functions, tangent hyperplanes, and lower hull. By this characterization the concept of higher-order Voronoi diagram is also obtained (Section 3) and the properties for Delaunay-type triangulation, defined in Section 4, can be shown.

The Voronoi diagram, which gives a proximate relation in the space, is a fundamental well-known object in computational geometry. This diagram in Euclidean space has been much studied [9, 20, 6, 11]. More specifically: the L_p metric [25]; the power diagram [9, pp. 327–328]; and the Laguerre Voronoi diagram [13]. There are, however, few investigations of Voronoi diagrams in Riemannian space. Voronoi diagrams on orbifolds, which represent surfaces (e.g., cylinders, Möbius bands, Klein bottles, flat toruses, and projective planes) are dealt with in [16]. Voronoi diagrams in hyperbolic space are treated in [5, 21]. There are some studies only a property of Voronoi diagram in Riemannian space: in manifolds without conjugate points [10], and Voronoi diagram on surface [18].

The concept of triangulation for given points has also been investigated ([4, 11]). The most famous useful triangulation is Delaunay triangulation, which has good properties in 2-dimensional Euclidean space [4]: it minimizes the maximum radius of a circumcircle, maximizes the minimum angle, and minimizes the maximum radius of an enclosing circle. The first and the second properties are proved by diagonal flip. The last is proved by using potential function for any number of dimensions ([26]). This triangulation is therefore often used in the finite element method, in mesh generation, and in computer graphics.

These structures are treated in a *dually flat space* with a *global coordinate system*, which is Riemannian space with two connections and in which two potential functions exist (Section 2). These

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potential functions are a natural extension of Euclidean $y = \sum_i (x_i)^2$ and can be used to compute some geometric structures efficiently. In Section 3, Voronoi diagram by *divergence* in dually flat space is proposed. This diagram is a subset of the lower envelope of tangent hyperplanes on the potential function as a face lattice. Therefore the structure of such a diagram is provided by removing some faces from the face lattice. We also described a higher-order Voronoi diagram in the space. The concept of Delaunay-type triangulation is dealt with in Section 4. Such a triangulation is defined by an algorithm which is the projection of the lower hull for lifting-up points on the potential function. This triangulation satisfies the following property: "the maximum minimum enclosing sphere for Delaunay-type triangulation minimize among all triangulations in dually flat space with global coordinate system." This result is an extension of [26].

2 Preliminaries

2.1 Statistical Parametric Space

Statistical parametric space is defined as a parametric space of probability density function, which has the close relation to the statistical estimation. In this section we explain their definitions, examples, characterizations and the relation to the statistical estimation.

2.1.1 Definitions and Examples of Statistical Parametric Space

Definition 2.1 (Statistical Parameter Space) *For a family of distributions over a domain which are determined by d real-valued parameters $[\xi_1, \ldots, \xi_d]$, the structure of this family is identified with $S = \{[\xi_1, \ldots, \xi_d] \mid [\xi_1, \ldots, \xi_d]$ corresponds one-to-one to a distribution in the family $\}$, which is called the* statistical parameter space *of this family of distributions.*

Here we define an example of a statistical parametric space, called *exponential family*, that contains the parametric space of normal distribution, Poisson distribution, etc.

Definition 2.2 (Exponential Family) *The* exponential family *is a family of probability distributions whose probability density function $p(x;\theta)$ (x is a probabilistic variable (vector in general) and θ is a parameter (vector in general) determining the distribution) is expressed as*

$$p(x;\theta) = \exp\left(C(x) + \sum_{i=1}^{d} \theta^i F_i(x) - \psi(\theta)\right)$$

for $\theta = [\theta^1, \ldots, \theta^d]$ and functions $C(x)$, $F_i(x)$ $(i = 1, \ldots, d)$ of x and a function $\psi(\theta)$ of θ.

Here, since $\int p(x;\theta)\mathrm{d}x = 1$, the function ψ for each θ is determined as follows:

$$\psi(\theta) = \log \int \exp\left(C(x) + \sum_{i=1}^{n} \theta^i F_i(x)\right) \mathrm{d}x. \tag{2.1}$$

Example 2.1 *1. For the one-dimensional normal distribution the probability density function with mean μ and standard deviation σ is $p(x;\theta)$ with*

$$\begin{aligned} &C(x) = 0,\ F_1(x) = x,\ F_2(x) = x^2, \\ &\theta^1 = \frac{\mu}{\sigma^2},\ \theta^2 = -\frac{1}{2\sigma^2}, \\ &\psi(\theta) = \frac{\mu^2}{2\sigma^2} + \log(\sqrt{2\pi}\sigma) = -\frac{(\theta^1)^2}{4\theta^2} + \frac{1}{2}\log\left(-\frac{\pi}{\theta^2}\right) \end{aligned}$$

2. For the one-dimensional Poisson distribution the probability density function is $\mathrm{e}^{-\xi}\frac{\xi^x}{x!}$ with probabilistic variable x on $\{0, 1, 2, \cdots\}$ with parameter space $\{\xi \mid \xi > 0\}$. For this distribution, $C(x) = -\log x!$, $F(x) = x$, $\theta = \log\xi$, $\psi(\theta) = \xi = \mathrm{e}^{\theta}$.

3. *For the distribution on a finite set* $\{x_0, x_1, \ldots, x_d\}$*, the probability that* x_i *occurs is* ξ_i *with parameter space* $\{[\xi_1, \ldots, \xi_d] \mid \xi_i > 0,\ \sum_{i=1}^{d} \xi_i < 1\}$*. For this distribution,* $C(x) = 0$*,* $F_i(x) = 1$ *when* $x = x_i$ *and 0 when* $x \neq x_i$*,* $\theta^i = \log \dfrac{\xi_i}{1 - \sum_{j=1}^{d} \xi_j}$ *and* $\psi(\theta) = \log \left(1 + \sum_{i=1}^{d} \exp \theta^i\right)$.

The defined Statistical parametric space, however, is only a parametric space of probability distribution. Since we want to regard this space as Riemannian space, we use a *metric* called the *Fisher information matrix*, which is famous in statistics. This idea was first considered in [27].

Definition 2.3 (Fisher information matrix) *For continuous probability density function* $p(x;\xi)$*, the matrix*

$$g_{ij} = \int \frac{\partial}{\partial \xi_i} \log p(x;\xi) \cdot \frac{\partial}{\partial \xi_j} \log p(x;\xi) \cdot p(x;\xi) \mathrm{d}x$$

is called the Fisher information matrix.

For a discrete probability density function, this matrix is defined like this:

$$g_{ij} = \sum_{x} \frac{\partial}{\partial \xi_i} \log p(x;\xi) \cdot \frac{\partial}{\partial \xi_j} \log p(x;\xi) \cdot p(x;\xi).$$

This matrix is related to the error of estimation [8], i.e., it gives a bound of the squared error of any unbiased estimator.

The Fisher information matrix can be regarded as a Riemannian metric

$$\mathrm{d}s^2 = (\mathrm{d}\xi_1, \ldots, \mathrm{d}\xi_d) \cdot \{g_{ij}\} \cdot \begin{pmatrix} \mathrm{d}\xi_1 \\ \vdots \\ \mathrm{d}\xi_d \end{pmatrix},$$

and with this metric $\mathrm{d}s^2$, called the *Fisher metric*, the statistical parameter space can be considered a Riemannian space $(R, g) = (\xi, \mathrm{d}s^2)$.

Definition 2.4 (α-Connection [3]) *Let* $R = \{p_\xi\}$ *be a d-dimensional statistical parametric space with Fisher metric* g*. For any real number* α *the following functions are defined:*

$$\left(\Gamma^{(\alpha)}_{ij,k}\right)_\xi = \int \left(\partial_i \partial_j \log p_\xi + \frac{1-\alpha}{2} \partial_i \log p_\xi \partial_j \log p_\xi\right) \cdot (\partial_k \log p_\xi) \cdot p_\xi \mathrm{d}x. \tag{2.2}$$

These functions are α-connection $\nabla^{(\alpha)}$ *on* R.

In particular, $\nabla^{(1)}$ *is called* exponential connection $\nabla^{(e)}$ *and* $\nabla^{(-1)}$ *is called* mixture connection $\nabla^{(m)}$.

It is possible that Levi-Civita connection [14] is defined in the statistical parametric space. That is an ordinary method in differential geometry. For example, the statistical parametric space of a normal distribution is equal to a Poincaré space as a Riemannian manifold if the Levi-Civita connection is induced. In [19], however, α-connection is induced because ∇-connection has a close relation to statistical estimation. That relation is described in Section 2.1.2. In this thesis the *α-connection* is used as a connection of such a parametric space.

The following lemmas about the exponential family were proved in [3].

Lemma 2.1 ([3]) *The exponential family is* $\nabla^{(e)}$*-flat.*

Lemma 2.2 ([3]) *The following is equivalent for a statistical parametric space* R *with Fisher metric* g*:*

- $\nabla^{(\alpha)}$*-connection is flat (* R *is* $\nabla^{(\alpha)}$*-flat).*
- $\nabla^{(-\alpha)}$*-connection is flat (* R *is* $\nabla^{(-\alpha)}$*-flat).*

If e- and m-connection are induced in the exponential family R with Fisher metric g, then that becomes *dually flat space*. There is an *affine coordinate system* in flat space. Since this space has two flat connections, there are two coordinate systems: the *natural parameter* $[\theta]$ and the *expectation parameter* $[\eta], \eta_i = \int F_i(x)p(x;\xi)dx$, which respectively correspond to the exponential and mixture connections.

2.1.2 Statistical Estimation

We state the relation between the maximum likelihood estimation and these connections. First we explain the *statistical estimation*, more details of which are in [1, 17]. Statistical estimation finds information of unknown probability function from observed value that follows the function. When let $x_{(1)}, \ldots, x_{(n)}$ be independent observed value of stochastic vector by a probability function, it is a statistical estimator to give the parameters ξ of original function $p(x;\xi)$ by $x^{(n)} = (x_{(1)}, \ldots, x_{(n)})$.

Definition 2.5 *A value of ξ is a* maximum likelihood estimator *if, for given x^n, the function $p_{(n)}(x^{(n)};\xi)$ is regarded as a function of only parameter ξ, which is called* likelihood function, *and maximize $p_{(n)}(x^{(n)};\xi)$ at $\tilde{\xi}$. That is*

$$\max_{\xi} p_{(n)}(x^{(n)};\xi) = p_{(n)}(x^{(n)};\tilde{\xi})$$

where $p_{(n)}(x^{(n)};\xi)$ is a probability density function of $x^{(n)}$. Such estimation is called maximum likelihood estimation.

For example, maximum likelihood estimation of the exponential family (Definition 2.2) is considered. Without loss of generality we can suppose $C(x) \equiv 0$ and $x_i := F_i(x)$. Then the probability density function becomes

$$p(x;\theta) = \exp\left(\sum_{i=1}^{d} \theta^i x_i - \psi(\theta)\right).$$

Since each x_i is independent, $p_{(n)}(x^{(n)};\theta)$ becomes

$$p_{(n)}(x^{(n)};\theta) = \prod_{i=1}^{n} p(x_{(i)};\theta) = \exp\left[\sum_{j=1}^{d} n\left\{\theta^j \bar{x}_j - \psi(\theta)\right\}\right] \tag{2.3}$$

where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_{(i)}. \tag{2.4}$$

In statistical parametric space with θ coordinate system if consider the problem that a maximized problem of equation (2.3). By differentiation we obtain

$$\bar{x}_i = \partial\psi(\theta) = \eta_i.$$

Therefore the maximum likelihood estimator is given in η-coordinate system of the statistical parametric space of the exponential family. That is, this estimation for exponential family is $\bar{x}$ and we need only compute its θ-coordinate expression $\theta = \theta(\eta(\bar{x}))$.

2.2 Dually Flat Space

This section describes *dually flat space* ([1, 3]), which is a kind of Riemannian space and includes *statistical parametric space* [1], *feasible region of linear programming* [30], *parametric family of invertible linear system* [2], etc. We define this space and give some examples. We also explain the two coordinate system, the potential functions and divergences in this space. These concepts are extensions of the corresponding concepts applicable to Euclidean space, and there is also an extension of the Pythagorean theorem (Theorem 2.1).

2.2.1 Definitions and Examples of Dually Flat Space

Definition 2.6 (Dually Flat Space) *Let (R, g) be a Riemannian space with two affine connections ∇, ∇^*. The 4-tuple (R, g, ∇, ∇^*) is a* dually flat space *if the Riemannian space satisfies the following conditions:*

- *For $\forall X, Y, Z$ vector field of R*

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla^*_Z Y\rangle. \tag{2.5}$$

  The connections ∇ and ∇^* are dual *about g if they satisfy this condition.*

- *∇ and ∇^* is flat.*

[**Remark**] If the $\nabla = \nabla^*$ then the condition (2.5) becomes the condition of metrical connection [14]. Thus this dual condition is a natural extension of metrical connection. In the other words, metrical connection is self-dual.

Example 2.2 (Euclidean Space) *Let $R = \mathbb{R}^d$ be a Riemannian space with a metric $g = I_d$. The connections ∇ and ∇^* are the same and self-dual. This space is called d-dimensional Euclidean space.*

Example 2.3 (Statistical Parametric Space of One-dimensional Normal Distributions) *Let $R = [\mu, \sigma](\sigma > 0)$ be a Riemannian space with Riemannian metric $g = \frac{1}{\sigma^2}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. This space is the parameter space of the normal distribution. That is, the normal distribution is given like this:*

$$p(x; \xi) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}. \tag{2.6}$$

The parametric space $[\xi] = [\mu, \sigma]$ has the metric which is called the Fisher information matrix (Definition 2.3). This space becomes dually flat space by inducing e*-connection and* m*-connection (Definition 2.4).*

The multinomial distribution is treated in [21].

Example 2.4 (Statistical Parametric Space of the Exponential Family) *The probability density function is already defined in Definition 2.2. Its Fisher information matrix becomes*

$$g_{ij} = \frac{\partial^2}{\partial\theta^i \partial\theta^j}\psi(\theta)$$

where $\psi(\theta)$ is given by equation (2.1). In this space exponential and mixture connections are also induced.

By a coordinate system $[\xi]$ the condition (2.5) becomes

$$\partial_k g_{ij} = \Gamma_{ki,j} + \Gamma^*_{kj,i}$$

where $\Gamma_{ij,k} = \sum_h \Gamma^h_{ij} g_{hk}$ and $\Gamma^*_{ij,k} = \sum_h \Gamma^{h*}_{ij} g_{hk}$. $\Gamma^k_{ij}, \Gamma^{k*}_{ij}$ are respectively the connection coefficients for ∇ and ∇^*.

Since the dually flat space is flat, there exist affine coordinate systems such that $\Gamma^i_{jk}, \Gamma^{i*}_{jk} \equiv 0$. The one for ∇-connection is called θ-coordinate, and the one for ∇^*-connection is called η-coordinate. These coordinates are called a *dual coordinate system* if this condition is satisfied:

$$\left\langle \frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial\eta_j} \right\rangle = \delta^j_i.$$

[**Remark**] In this paper we use the following notations: $\partial_i = \frac{\partial}{\partial\theta^i}, \partial^j = \frac{\partial}{\partial\eta_j}$.

Example 2.5 (Dual Coordinates) *1. For Euclidean space the dual coordinate systems are $\theta^i = \eta_i$.*

2. *For the parametric space of one-dimensional normal distributions the dual coordinates systems become*

$$[\theta^1, \theta^2] = \left[\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}\right], \quad [\eta_1, \eta_2] = \left[\mu, \mu^2 + \sigma^2\right].$$

3. *For the parametric space of the exponential family the dual coordinates systems become*

$$[\theta^1, \ldots, \theta^d], \ \eta_i = \int F_i(x) p(x;\theta) \mathrm{d}x,$$

these coordinate systems are called natural parameter *and* expectation parameter*, respectively.*

Let $\psi, \varphi : R \to \mathbb{R}$ be functions such that

$$\partial_i \psi = \eta_i, \quad \partial^j \varphi = \theta^j. \tag{2.7}$$

Those differential equations can be solved if $\partial_i \eta_j = \partial_j \eta_i, \partial^i \theta^j = \partial^j \theta^i$. In this paper we suppose these conditions. By this assumption the differential equations above have solutions. Moreover these functions are convex in Euclidean space because the quadratic differentiations of these functions is the metric of Riemannian space and the metric is a positive definite matrix.

Thus the following equations are proved:

$$\partial_i \partial_j \psi = g_{ij}, \quad \partial^i \partial^j \psi = g^{ij}, \tag{2.8}$$

$$\varphi + \psi - \sum_i \theta^i \eta_i = 0, \tag{2.9}$$

where g_{ij} is an expression of g in the θ-coordinate system and g^{ij} is in the η-coordinate system. The transformation defined by equation (2.7) and (2.9) is a *Legendre transformation* and the functions ψ and φ are called *potential functions.*

Example 2.6 (Potential Functions) *1. For Euclidean space the potential functions become*

$$\psi = \varphi = \frac{1}{2} \sum (\theta^i)^2.$$

Therefore it is also stated that Euclidean space is self-dual.

2. *For the parametric space of normal distribution, the potential functions become*

$$\begin{aligned} \psi &= \frac{\mu^2}{2\sigma^2} + \log(\sqrt{2\pi}\sigma) &&= -\frac{(\theta^1)^2}{4\theta^2} + \frac{1}{2}\log\left(-\frac{\pi}{\theta^2}\right), \\ \varphi &= -\log(\sqrt{2\pi}\sigma) - \frac{1}{2}. \end{aligned}$$

3. *For the statistical parametric space of the exponential family, the potential functions are given by equation (2.1) and become*

$$\varphi(\theta) = \int (\log p(x;\theta) - C(x)) \cdot p(x;\theta) \mathrm{d}x.$$

This function φ is an entropy [8] if $C(x) \equiv 0$.

In the dually flat space *divergence* can be defined, which is a distance-like function. In Section 3.2 we use this divergence instead of distance.

Definition 2.7 (Divergence) *By these dual coordinate systems and potential functions a mapping $D : R \times R \to \mathbb{R}$ is defined as follows:*

$$D(\theta \| \tilde{\theta}) = \psi(\theta) + \varphi(\tilde{\theta}) - \sum_{i=1}^{d} \theta^i(\theta) \eta_i(\tilde{\theta}). \tag{2.10}$$

This function is called ∇-divergence *and is distance-like function in the dually flat space. Another divergence, called* ∇^*-divergence, *is defined:*

$$D^*(\theta\|\tilde{\theta}) = \psi(\tilde{\theta}) + \varphi(\theta) - \sum_{i=1}^{d} \theta^i(\tilde{\theta})\eta_i(\theta). \tag{2.11}$$

The divergence does not satisfy the symmetry of the distance, but the value of this function is non-negative and vanishes if and only if two points are the same.

[**Remark**] By this definition we can get the following property of divergence

$$D^*(\theta\|\tilde{\theta}) = D(\tilde{\theta}\|\theta). \tag{2.12}$$

Example 2.7 (∇- and ∇^*-divergence) *1. Euclidean space [1], the* ∇- *and* ∇^*-*divergences are the same.*

$$D(p\|q) = D^*(p\|q) = \frac{1}{2}\sum_{i=1}^{d}\left\{\theta^i(p) - \theta^i(q)\right\}^2.$$

This divergence is a half of the square of Euclidean distance.

2. *Statistical parametric space of one-dimensional normal distribution [22], the* ∇- *and* ∇^*-*divergences are expressed as follows*

$$D(\xi^{(2)}||\xi^{(1)}) = D^*(\xi^{(1)}||\xi^{(2)}) = \log\frac{\sigma^{(2)}}{\sigma^{(1)}} + \frac{\left(\sigma^{(1)}\right)^2 + \left(\mu^{(1)} - \mu^{(2)}\right)^2}{2(\sigma^{(2)})^2} - \frac{1}{2} \tag{2.13}$$

where $\xi^{(1)} = [\mu^{(1)}, \sigma^{(1)}], \xi^{(2)} = [\mu^{(2)}, \sigma^{(2)}]$. *This divergence is the calculation of* Kullback-Leibler divergence [8] *for the normal distribution.*

3. *Statistical parametric space of exponential family, the* ∇- *and* ∇^*-*divergences are expressed as follows:*

$$D(p\|q) = D^*(q\|p) = \int q\log\frac{q}{p}dx.$$

The ∇^*-*divergence is* Kullback-Leibler divergence, *which is famous and useful in information theory* [8].

For three points in the dually flat space, a Pythagorean theorem like a Euclidean space is proved.

Theorem 2.1 (Pythagorean theorem [1]) *Suppose three points* p, q, r *in the dually flat space. If the* ∇-*geodesic connecting* p, q *and the* ∇^*-*geodesic connecting* q, r *intersect orthogonally at* q, *then*

$$D(p\|r) = D(p\|q) + D(q\|r).$$

This theorem is an extension of Pythagorean theorem in Euclidean space.

We define a *global coordinate system.*

Definition 2.8 (Global Coordinate System) *An affine coordinate system is* global *if the same coordinate system is used for all points within a given space. Such a coordinate system is called a* global coordinate system.

If a flat space has a global coordinate system, then a geodesic becomes a Euclidean line in that coordinate system.

3 Voronoi Diagram

The aim of this section is to discuss the proximity relation in Riemannian space, leading to the *Voronoi diagram.* That is defined in general Riemannian space. Since our aim is to construct a geometric structure, the dually flat space is treated and we propose an efficient algorithm.

3.1 Voronoi Diagram in Riemannian Space

The Voronoi diagram is a well-known geometric and mathematical object because it is simple to define and has applications in a number of sciences: biology, physics, archaeology, etc. First we provide a general definition.

Definition 3.1 *Let $P = \{p^{(i)} \ ; \ i = 1, \ldots, n\}$ be a set of n points in the space S. The* Voronoi polygon $\mathrm{Vor}(p^{(i)})$ *of P is defined as follows:*

$$\mathrm{Vor}(p^{(i)}) = \left\{ x \in S \mid d(x, p^{(i)}) \leq d(x, p^{(j)}) \ \forall j \neq i \right\} \tag{3.1}$$

where $d : S \times S \to \mathbb{R}$ is a distance function. The Voronoi polygons for P partition S and constitute the Voronoi diagram. *The vertices of the Voronoi polygons are called* Voronoi points *and the k-dimensional faces of the Voronoi polygons are called* Voronoi k-faces.

This definition uses proximity relations between points in the space, based on a certain distance function. There are different criteria used in the literature [20], such as the L_p metric [25], the power diagram (see [9, pp.327–328]), the Laguerre Voronoi diagram [13], and the Voronoi diagram in hyperbolic space [5, 21].

The original criterion of the Voronoi diagram is Euclidean distance and with regard to that criterion there has been much research and consequently so interesting results: the relation to hyperplane arrangement, to Delaunay triangulation, etc. The Voronoi diagrams dealt with in Section 3.2, called ∇- and ∇^*-Voronoi diagrams, also have these properties, That is the dually flat space is an extension of Euclidean. These diagrams are also constructed by the projection of the lower envelope of a hyperplane arrangement and the deletion of some faces (Theorem 3.1, Theorem 3.2). Moreover, the relation between hyperbolic, ∇-, and ∇^*-Voronoi diagrams are specified in Section 3.3, where we describe the relation between Fisher metric and divergence, and obtain the characterization of Voronoi diagram in Poincaré space and statistical parametric space of one-dimensional normal distributions.

3.2 Dually Flat Space

This section deals with Voronoi diagrams in the dually flat space, is called ∇-Voronoi diagram and ∇^*-Voronoi diagram. The dually flat space was already described in Section 2.2. In that space, like in a Euclidean space, there are *potential functions* and *divergence*. In addition, since the relation between potential function and distance can be also shown, some properties of potential function are proved. This work is an extension of [22, 23].

3.2.1 Voronoi Diagrams in Dually Flat Space

Firstly, these diagrams are defined by ∇-divergence and ∇^*-divergence instead of distance.

Definition 3.2 (∇-Voronoi Diagram) *For a set of n points $\{\theta^{(j)} \ ; \ j = 1, \ldots, n\}$ in the dually flat space, the Voronoi region $V(\theta^{(j)})$ of $\theta^{(j)}$ is defined by*

$$V(\theta^{(j)}) = \bigcap_{k \neq j} \left\{ \theta \ \middle| \ D(\theta \| \theta^{(j)}) < D(\theta \| \theta^{(k)}) \right\}.$$

$V(\theta^{(j)})(j = 1, \cdots, n)$ portions the dually flat space R, which is called the ∇-Voronoi diagram of these points.

Definition 3.3 (∇^*-Voronoi diagram) *The ∇^*-Voronoi diagram is defined by replacing*

$$D(\theta \| \theta^{(j)}) < D(\theta \| \theta^{(k)})$$

in the definition of the ∇-Voronoi diagram with

$$D^*(\theta \| \theta^{(j)}) < D^*(\theta \| \theta^{(k)}). \tag{3.2}$$

[**Remark**] Because the divergence does not satisfies the symmetry of the distance, two different Voronoi diagrams are defined in the dually flat space. However there is the relation (2.12), however, we can rewrite equation (3.2) using ∇-divergence:

$$D(\theta^{(j)}\|\theta) < D(\theta^{(k)}\|\theta).$$

Thus two Voronoi diagrams are defined, but the difference between these only is the direction of the divergence: from a generator to a point or from a point to a generator. Moreover the boundary of the $\nabla(\nabla^*)$-Voronoi region consists of the $\nabla(\nabla^*)$-geodesic segments, respectively.

The ∇- and ∇^*-Voronoi diagrams for a set of points might have the same structure as a face lattice but sometime their structure are different.

Example 3.1 ∇- and ∇^*-Voronoi diagrams for a set of points do not always have the same structure. For example, let $P = \{(-a,1),(0,b),(a,1) \mid a,b > 0,\ \sqrt{a^2+1} < b \leq e^a\}$ be a set of points in the statistical parametric space of one-dimensional normal distributions. In the ∇-Voronoi diagram for P there is a point equidistant from three points, but in the ∇^*-Voronoi diagram for P there is no such point (Fig. 1).

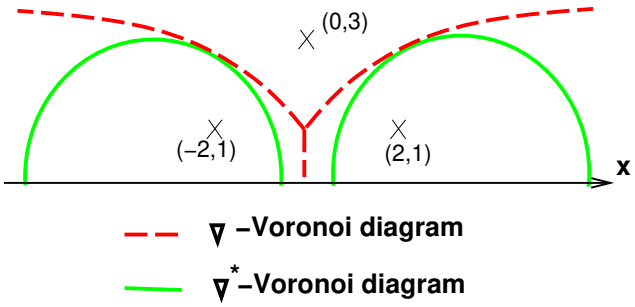

Figure 1: ∇- and ∇^*-Voronoi diagrams for P

In this section we suppose that a dually flat space has a *global coordinate system* because unless there is such a coordinate system, we specify only a local property of a geometric structure.

Lemma 3.1 *The following conditions are equivalent:*

- *there exists a global coordinate system,*
- *there exists a unique potential functions. That is, the function is defined over the entire space.*

Proof: Relations (2.7)specify the relation between the coordinate system and the potential function. In general, these relations are proved only in the local coordinate system. But if the coordinate system is global, this relation between the coordinate system and the potential function is expanded to the whole space, i.e., a unique potential function in the coordinate system is defined.

Conversely, if such a potential function exists, then its derivative becomes the global coordinate system. □

Example 3.2 *The statistical parametric space has a* global coordinate system. *Since the potential function in the parametric space is actually written, it is proved by the above lemma that the space has a global coordinate system.*

3.2.2 ∇-Voronoi Diagram

In this section we specify the structure of the ∇-Voronoi diagram in the dually flat space R for n given point $\theta^{(j)}$. We also give an algorithm for the construction of the ∇-Voronoi diagram.

We suppose an additional axis y to R so that we can consider a space $[R, y]$ in which we consider n hypersurfaces

$$y = D(\theta \| \theta^{(j)}) \quad (j = 1, \cdots, n).$$

Lemma 3.2 *Suppose a point $\tilde{\theta}$, the potential function ψ in the dually flat space R, and a space $[R, y] = [\theta, y]$. The difference between $-\psi(\theta)$ and the value of the y-coordinate at θ in the tangent hyperplane of the potential function ψ at $\tilde{\theta}$ is equal to the ∇-divergence from θ to $\tilde{\theta}$.*

Proof: The tangent hyperplane is expressed as

$$y = -\sum_i \left(\theta^i - \tilde{\theta}^i\right) \frac{\partial \psi(\tilde{\theta})}{\partial \theta^i} - \psi(\tilde{\theta}) = -\sum_i \left(\theta^i - \tilde{\theta}^i\right) \eta_i(\tilde{\theta}) - \psi(\tilde{\theta}).$$

Because the potential function is strictly convex and the tangent hyperplane is above the function, the difference becomes

$$\begin{aligned} -\sum_{i=1}^{d} (\theta^i - \tilde{\theta}^i)\eta_i(\tilde{\theta}) - \psi(\tilde{\theta}) - (-\psi(\tilde{\theta})) &= \psi(\theta) + \varphi(\tilde{\theta}) - \sum_{i=1}^{d} \theta^i \eta_i(\tilde{\theta}) \\ &= D(\theta \| \tilde{\theta}). \end{aligned}$$

□

[**Remark**] This equation can also be proved in the Euclidean space. Recall the example of ∇- and ∇^*-divergence in Euclidean space, and the difference between the potential function and the tangent hyperplane becomes half of the square of the Euclidean distance. This fact to show the relation between hyperplane arrangement and the Euclidean Voronoi diagram was used in [9, 25].

By the theory of Voronoi diagrams (e.g., see [9]), the projection of the lower envelope of that hyperplane arrangement becomes a Euclidean Voronoi diagram. In the dually flat space the projection of the lower envelope of these hypersurfaces to the space R becomes the ∇-Voronoi diagram. In the θ-coordinate system suppose a mapping from $[R, y]$ to itself by $[\theta, y] \mapsto [\theta, y - \psi]$. The hypersurface is mapped to

$$y = -\left(\sum_{i=1}^{d} \theta^i \eta_i(\theta^{(j)}) - \varphi(\theta^{(j)})\right).$$

This image of the hypersurface is the *hyperplane* in the space $[\theta, y]$. Because the $\theta^{(j)}$ is a generator and constant, the $\varphi(\theta^{(j)})$ is also constant. Moreover the map retains the above-below relation with respect to the y-direction.

So we state next theorem.

Theorem 3.1 *For a set of n points $\{\theta^{(j)} \ ; \ j = 1, \ldots, n\}$ in the d-dimensional dually flat space with global coordinate system, a ∇-Voronoi diagram is the projection of the lower envelope of n hyperplanes*

$$y = -\left(\sum_{i=1}^{d} \theta^i \eta_i(\theta^{(j)}) - \varphi(\theta^{(j)})\right) \tag{3.3}$$

$(j = 1, \ldots, n)$ to the dually flat space R which is a subset of d-dimensional Euclidean space as a face lattice and hence the combinatorial complexity F is $O(n^{\lceil (d+1)/2 \rceil})$ and the time complexity is $O(F \log n)$.

Proof: The projection to R means $[\theta, y] \mapsto [\theta]$. As mentioned above, the original lifting-up map keeps the above-below relation is unchanged, and for the hyperplane for $\theta^{(j)}$ attaining the lower envelope at $[\theta]$,

$$\min_k D(\theta \| \theta^{(k)}) = D(\theta \| \theta^{(j)}).$$

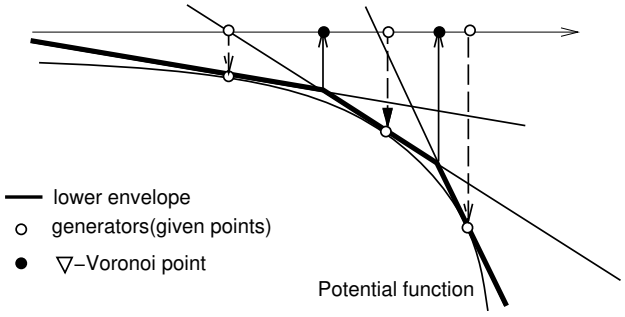

Figure 2: Potential function and the tangent hyperplane at each generator

To derive the combinatorial complexity, we apply the upper bound theorem for convex polytopes with n facets in the $(d+1)$-dimensional Euclidean space. To derive the time complexity, we can use any convex hull algorithm in computational geometry. □

By this theorem we have the following corollary.

Corollary 3.1 *Each Voronoi region of the ∇-Voronoi diagram is nonempty and it is a convex polyhedron in the θ-coordinate system, ignoring the part at infinity.*

Proof: Since each of Voronoi region is included in the generator, the space is nonempty. Moreover, in the θ-coordinate system, all of Voronoi regions consist of half-spaces of hyperplane. Thus the Voronoi region is convex and the polyhedron in the θ-coordinate system. □

The upper envelope, whose projection becomes the farthest ∇-Voronoi diagram, which is naturally defined, can be obtained. The higher-order Voronoi diagram is also defined in the space by ∇-divergence as same as in Euclidean. We can get the relation between the hyperplane arrangement and all higher-order ∇-Voronoi diagrams.

Corollary 3.2 (Higher-order ∇-Voronoi Diagram) *The hyperplane arrangement by equation (3.3) has the complete information of all higher-order ∇-Voronoi diagrams. In other words, the face lattice of the all higher-order ∇-Voronoi diagrams and of the hyperplane arrangement in the θ-coordinate system in the above theorem is the same except for some faces which is not included in the diagram.*

Proof: This corollary can be proved easily because by Lemma 3.2 and Theorem 3.1, the k-level of that hyperplane arrangement corresponds to the kth-order ∇-Voronoi diagram.

It should be noted that the arrangement of hyperplanes in the theorem is defined in the whole θ-coordinate system $[\theta]$, regarding it as $\mathbb{R}^d$. The ∇-Voronoi diagram, however, is defined on $\theta(R)$, which may not be the whole $\mathbb{R}^d$, and may be a proper subset. In such a case, part $\mathbb{R}^d - \theta(R)$ should be cut out, via the projection to R stated in the theorem. □

Moreover, the complexity of the kth-order ∇-Voronoi diagram is bounded.

Corollary 3.3 *For a set of n points in the d-dimensional dually flat space with a global coordinate system, the combinatorial complexity of the kth-order ∇-Voronoi diagram is $O(k(n-k))$ when $d=2$ and is $O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil + 1})$ when $d \geq 3$.*

Proof: The face lattice of the kth-order ∇-Voronoi diagram corresponds to the k-level of above hyperplane arrangement (Corollary 3.2). Thus its combinatorial complexity is bounded by the k-level of the hyperplane arrangement when $d \geq 3$ [7].

A proof of the bound on combinatorial complexity when $d=2$ given in [15, 9]. The essence of that proof is based on the following:

- the correspondence between the kth-order Voronoi diagram and the k-level of the hyperplane arrangement;

- the graph of the upper(lower) part of any cell in the k-level is connected, planar and no cycles without the upper graph of the topmost cell and the lower graph of the bottommost cell;

  and

- the Euler relation for planar graph.

So the hyperplane arrangement (3.3) also satisfies conditions above, because the first is already stated and the second is proved similarly as in Euclidean. In addition, since the hyperplane arrangement is regarded as Euclidean, the Euler relation can be used.

More specifically, the analysis in [9] can be applied to the hyperplane arrangement for n tangent hyperplanes of a strictly convex function. □

The following algorithm can be used to generate the ∇-Voronoi diagram in the dually flat space.

Algorithm (∇-Voronoi diagram)

1. Map the given set of points to the θ-coordinate system.
2. Construct the Voronoi diagram for such points by using the potential function ψ,
   i.e, by computing a tangent hyperplane arrangement of the transformed set and projecting the arrangement to θ-space.
3. Remove faces which are not included in ∇-Voronoi diagram.

Steps 1 and 2 are already explained however in Step 3 we decide which face is included in ∇-Voronoi diagram. That face can be determined by the following theorem.

Theorem 3.2 *In the space $[R, y] = [\eta, y]$, consider the lower boundary of the convex hull of n points $[-\eta^{(j)}, -\psi(\eta^{(j)})]$ $(j = 1, \ldots, n)$ by regarding this space as the $(d+1)$-dimensional Euclidean space. In the face lattice of the lower boundary of this convex hull, delete each face which does not have a supporting hyperplane at the face (i.e., hyperplane containing this face and all other points not in this face are located in one proper side of the hyperplane) given by $-\sum_{i=1}^{d} \eta_i \theta^i + c$ for $\theta = \theta(q)$ with some $q \in R$ and a constant c.*

Then the sublattice consisting of the remaining faces on the lower boundary of the convex hull is dual to the lattice of the ∇-Voronoi diagram.

Proof: This is basically the point-hyperplane duality sometimes called polarity (e.g., see [9, 25]). That is the hyperplanes (3.3) are regarded as the points in the η-coordinate system. Since this polarity keeps the above-below relation unchanged, the lower part of the hyperplane arrangement maps the lower part of the convex hull of those points. Thus the lower boundary of the convex hull in the dual space contains all the information of the ∇-Voronoi diagram. However some part of this ∇-Voronoi diagram, i.e., if the supporting hyperplane of such face does not satisfy the above equation, such face is not included in the ∇-Voronoi diagram. □

So we define *unbounded region* in the dually flat space and the hyperbolic space.

Definition 3.4 *Let R be a region in the dually flat space (hyperbolic space). That region is* unbounded *if when a point p in R and a direction are given, then the geodesic from p with that direction is contained in R and for any point q on the geodesic there exists a point r on the geodesic s.t. $D(p\|q) < D(p\|r)$ $(d(p,q) < d(p,r))$, respectively. In other words, such a geodesic can be extended infinitely in the direction.*

Example 3.3 *In the statistical parametric space of the normal distribution (Example 2.1) there is an* infinite boundary. *That is, in the η-coordinate system $R = \{(\eta_1, \eta_2) \| \eta_2 > {\eta_1}^2\}$ and its boundary becomes*

$$\eta_2 = {\eta_1}^2.$$

So the divergence from any point to boundary point increases (see equation (2.13)).

Lemma 3.3 *If a region in the dually flat space with global coordinate system has a part of the infinite boundary, then the region is unbounded.*

Proof: In that space there is a potential function s.t. that function is defined only in the space and its value at a point on the infinite boundary diverges. The divergence is the difference between potential function and its conjugate function (Definition 2.7). Therefore the value of divergence also diverges at such a point. So suppose a point in the region with a part of infinite boundary and a direction to the infinite boundary, then the geodesic from that point can be extended. Thus that region is unbounded. □

We next give a theorem characterizing unbounded Voronoi regions.

Theorem 3.3 *1. In Theorem 3.2, in the space $[R, y] = [\eta, y]$, consider the face lattice of the whole lower boundary of the ∇^*-convex hull of n points $[-\eta^{(j)}, -\varphi(\eta^{(j)})]$ $(j = 1, \ldots, n)$.*

Then the Voronoi region of $\theta^{(j)}$ is unbounded in R iff there exists a sequence of points $\theta^{(l)} \in R$ diverging to infinity in R and constants $c^{(l)}$ such that $-\sum_{i=1}^{d} \theta^{(l)i}\eta_i^{(j)} + c^{(l)}$ is a supporting hyperplane to the hull at $[-\eta^{(j)}, -\varphi]$.

2. If $\eta^{(j)}$ is on the boundary of the ∇^*-convex hull of $\eta^{(k)}$ $(k = 1, \ldots, n)$, the Voronoi region of $\theta^{(j)}$ is unbounded in R.

Proof: It should be noted that the boundary of the space is regarded as being at infinity, i.e., the divergence from any point in the space to a boundary point diverges to infinity. These results are based on the duality between θ- and η- coordinate systems. Then (1) is basically characterizes unbounded Voronoi regions in the dual setting.

As for (2), $\theta^{(j)}$ has an unbounded region when the lower hull in Theorem 3.2 is projected to $[\theta]$ regarded as $\mathbb{R}^d$. Then, restricting $\mathbb{R}^d$ to $\theta(R)$, it remains unbounded. □

It should be noted that the converse of (2) in this theorem is not necessarily true.

We now compare the ∇-Voronoi diagram with the Euclidean Voronoi diagram. For n points $\xi^{(j)}$ $(j = 1, \ldots, n)$ in $\mathbb{R}^d$, by considering an additional axis y, the Euclidean Voronoi diagram is the projection to the original space of the lower envelope of the tangent hyperplanes at $\xi^{(j)}$ to a parabolic surface $y = \|\xi\|^2$, where $\|\cdot\|$ is the L_2 norm. Its dual structure, the Delaunay triangulation, is the projection to the original space of the lower convex hull of points $(\xi^{(j)}, \|\xi^{(j)}\|^2)$.

Thus, by making correspondence between $y = \psi(\theta)$ and $y = \|\xi\|^2$ and recalling $\partial^i\varphi = \theta^i$ and $\varphi + \psi = \sum_{i=1}^{d} \theta^i\eta_i$, everything corresponds to each other, except that in the ∇-Voronoi diagram the part of the lower envelope of $\mathbb{R}^d - \theta(R)$ is meaningless.

This may be viewed as a computational-geometric interpretation of the dual connection and dually flat space formed by the Kullback-Leibler divergence [1, 3], or we should say that this is an implication of the Legendre transformation.

3.2.3 ∇^*-Voronoi Diagram

In this section we investigate the ∇^*-Voronoi diagram on the manifold R for n given points in the dually flat space. By the duality between θ and η, everything carries over when their roles are interchanged.

By considering the map

$$[\eta, y] \mapsto [\eta, y - \varphi]$$

and hyperplanes $y = -\left(\sum_{i=1}^{d} \theta^i(\tilde{\eta})\eta_i - \psi(\tilde{\eta})\right)$, we can rewrite theorem 3.1 for the ∇^*-Voronoi diagram.

Theorem 3.4 *For a set of n points $\{\eta^{(j)} \ ; \ j = 1, \ldots, n\}$ in the d-dimensional dually flat space with global coordinate system, the ∇^*-Voronoi diagram is the projection of the lower envelope of n hyperplanes*

$$y = -\left(\sum_{i=1}^{d} \theta^i(\eta^{(j)})\eta_i - \psi(\eta^{(j)})\right) \tag{3.4}$$

$(j = 1, \ldots, n)$ to the dually flat space R which is a subset of d-dimensional Euclidean space as a face lattice and hence the combinatorial complexity F is $O(n^{\lceil (d+1)/2 \rceil})$ and the time complexity is $O(F \log n)$.

Other corresponding theorems can be obtained similarly. In fact, from the viewpoint of convex analysis, these duality results can all be regarded as results of the Legendre transform between θ- and η-coordinate systems. Since we are treating a finite set of 'points', these dualities correspond completely to the well-known duality results in computational geometry.

In the dually flat space, the ∇- and ∇^*-Voronoi diagrams are characterized by the potential functions ψ and φ, respectively. Thus these diagram is based on the potential functions. Suppose a twice differentiable and strictly convex function $\psi(\theta)$ on the space R. A discussion similar to the one above applies: the divergence is defined by ψ, and the *conjugate function* φ of ψ and the η-coordinate system are defined like this :

$$[\eta_i] := \frac{\partial \psi}{\partial \theta^i}, \quad [\eta] = \left\{ \frac{\partial \psi(\theta)}{\partial \theta}; \theta \in R \right\}.$$

Therefore these theories can be considered to be a kind of convex analysis ([28]).

3.3 Relation of Hyperbolic, ∇- and ∇^*-Voronoi Diagram

The previous section described the ∇ and ∇^*-Voronoi diagrams. Each structure is independently defined, but there is a relation between hyperbolic Voronoi diagram [21], ∇-Voronoi diagram, and ∇^*-Voronoi diagram.

We first deal with the relation between the metric and the divergence.

Lemma 3.4 *The infinitesimal change of divergence approximates the sum of the Fisher metric ignoring the higher-order part That is,*

$$D(\theta + d\theta \| \theta) = \sum_{i,j} g_{ij} d\theta^i d\theta^j + O((d\theta)^3).$$

Proof: By expanding $D(\theta + d\theta \| \theta)$ at θ, we get

$$\begin{aligned} D(\theta + d\theta \| \theta) &= \sum_{i,j} \partial_i \partial_j \psi(\theta) \mathrm{d}\theta^i d\theta_j + O((d\theta)^3) \\ &= \sum_{i,j} g_{ij} d\theta^i d\theta^j + O((d\theta)^3). \end{aligned}$$

□

Thus we get the relation that the infinitesimal change of ∇-divergence equals the sum of the metric and the change of the third order. Moreover the third order is used as a *connection* in the dually flat space.

For the ∇^*-divergence the following relation is also proved:

$$D^*(\eta + d\eta \| \eta) = \sum_{i,j} g^{ij} d\eta_i d\eta_j + O((d\eta)^3).$$

[**Remark**] This relation for Kullback-Leibler divergence is already shown in [27].

Now we explain the relation between the parametric space of one-dimensional normal distributions with Levi-Civita connection [21] and with dual connection. We consider n one-dimensional normal distributions $\theta^{(j)}$ with $[\mu^{(j)}, \sigma^{(j)}]$ $(j = 1, \ldots, n)$. We simply call the Voronoi diagram by Fisher metric with the Levi-Civita connection *hyperbolic Voronoi diagram* [21].

In the case of Euclidean Voronoi diagrams of points, unbounded Voronoi regions are characterized by the convex hull of given points. That is, the Voronoi region of a point is unbounded iff it is on the boundary of the convex hull (e.g., see [9, 25]). As we have seen so far, the ∇-Voronoi, ∇^*-Voronoi, and hyperbolic Voronoi diagrams all have polytopal structure, where the convexity plays a crucial role. In connection with the convexity, we analyze unbounded Voronoi regions in these diagrams and compare their properties.

We consider points with the largest σ value among given points in the $[\mu, \sigma]$ upper-half plane, and we show that the ∇^*-Voronoi and hyperbolic Voronoi diagrams share similar structure for these points because their geodesic is regarded as the same. Let

$$P_{\max} = \left\{ p^{(j)} \mid \sigma^{(j)} = \max_k \sigma^{(k)} \right\}.$$

Theorem 3.5 *In both the ∇^*-Voronoi and hyperbolic Voronoi diagrams, the following hold.*

1. *When $|P_{\max}| = 1$, for any σ and $\mu = \pm M$ with sufficiently large $M > 0$, a point $[\mu, \sigma]$ belongs to the Voronoi region of $p^{(j)} \in P_{\max}$.*

2. *Otherwise, $|P_{\max}| > 1$. For any σ and $\mu = M$ ($\mu = -M$, respectively) for sufficiently large $M > 0$, $[\mu, \sigma]$ belongs to the Voronoi region of $p^{(j)} \in P_{\max}$ with $\mu^{(j)}$ minimum (maximum, respectively). For sufficiently large σ, the Voronoi regions are ordered in the increasing order of $\mu^{(j)}$ ($p^{(j)} \in P_{\max}$) with Voronoi edge parallel to the σ-axis in the $[\mu, \sigma]$-plane.*

Proof: Voronoi edges are upper circular arcs in both cases, and then checking the equation of hyperbolic space and the parametric space of one-dimensional normal distribution, we obtain the theorem. □

Thus, at infinity except on the μ-axis in the $[\mu, \sigma]$-plane, the ∇^*-Voronoi and hyperbolic Voronoi diagrams have the same structure.

With regard to the μ-axis at infinity, for unbounded Voronoi regions in ∇^*-Voronoi diagram, Theorem 3.3 shows that the convex hull in the η-coordinate system provides a subset of points whose Voronoi regions are unbounded (it also gives a complete characterization for the unboundedness, which cannot be interpreted directly from the convexity).

We previously defined the convexity and the convex hull in Poincaré space [21], but this convex hull does not characterize unbounded Voronoi regions in the hyperbolic Voronoi diagram. For example, there is a case in which points in $P_{\max}$ characterized in Theorem 3.5 are not on the boundary of this convex hull. Thus, situation here is very different from that in the Euclidean case. Using results of Onishi and Takayama [24], we have for this case the following theorem corresponding to Theorem 3.3.

Theorem 3.6 *1. Consider the Delaunay triangulation of given points in $\mathbb{H}$ in the ordinary Euclidean sense. Delete each edge of the Delaunay triangulation such that there is no circle which passes the two endpoints of the edge, does not contain any given point in its inside, and is contained in $\mathbb{H}$. The remaining skeleton is connected, and, the Voronoi regions of points incident to the outer-face in this skeleton are unbounded.*

2. *The Voronoi region of a point on the boundary of a convex hull of given points in the ordinary Euclidean manner is unbounded.*

Proof: (1) That skeleton becomes a *Delaunay graph* [24]. The upper edge and interior part of the skeleton is equal to a Delaunay graph because the circumscribed circle is for the triangle of Delaunay triangulation in the ordinary Euclidean manner. Moreover, the deletion of the lower edge of the skeleton and the removed edge of the Delaunay graph are the identity.

When the lower edge of skeleton is removed, the edge is the removed edge of the Delaunay graph [24].

Conversely, suppose an deleted edge, whose end points are $p^{(1)}, p^{(2)}$ and a triangulation which contains that edge on the boundary. Let the rest point of the triangle be p. This triangle is deleted and the lower edge is also erased. We show that the edge is the removed edge of the skeleton. The centers of circles through $p^{(1)}, p^{(2)}$ are on the perpendicular bisector l of the segment. If that circle contains no other given points, then p also is not included. Therefore the center is on l and nearer to the boundary $\sigma = 0$ of the space than is the center of the Euclidean circumscribed circle for the triangle. Thus the circumscribed circle becomes large and is not contained in $\mathbb{H}$. Consequently, there is no circle with the condition.

The Delaunay graph is the dual structure of hyperbolic Voronoi diagram and Voronoi region always contacts with other region, then the connectivity of skelton is proved.

In addition, the Voronoi region of a point incident to the outer-face in the skeleton is unbounded for upper or intersects with the boundary of the space. In the former case the geodesic can be extended to the upper direction. In the latter case, if the distance is regarded as the parameter, then the length of the geodesic is infinite because of the metric. Thus if the region is tangent to $\sigma = 0$, it is unbounded.

The geodesic segment of the half-circle type that does not cross the other geodesic always comes near $\sigma = 0$. The boundary (geodesic segment) of such a Voronoi region intersects at $\sigma = 0$ and its region becomes unbounded.

(2) is basically implied by (1). □

Thus, in the ∇^*-Voronoi diagram case the convex hull in the η-coordinate system provides partial information for unbounded Voronoi regions, whereas in the hyperbolic Voronoi diagram case the convex hull in the original $[\mu, \sigma]$-plane does. In both cases, to completely characterize the unbounded Voronoi regions, some faces are removed from the convex hull. For this, the hyperbolic Voronoi case provides more intrinsic features, and it would be required to obtain such characterizations in the ∇^*-Voronoi diagram case.

4 Delaunay-type Triangulation

This section describes special triangulations, called *Delaunay* and *Delaunay-type* triangulations. These triangulations are useful in the finite element method, mesh generation, etc. We discuss their properties and construction, and we explain, the relations between these triangulations and Voronoi diagrams.

4.1 Dually Flat Space

In this section we obtain *Delaunay-type triangulation* in dually flat space. When we described the convex hull [21] and the Voronoi diagram (Section 3.2), we supposed the existence of a global coordinate system. Since convex hull and Voronoi diagram are used in this section, we also suppose the existence of a global coordinate system. The convex hull for a set of points in dually flat space is equal to the Euclidean convex hull [21]. Triangulation in dually flat space is defined in the same way it is defined in Euclidean space.

Moreover, Delaunay triangulation is also defined as follows:

Definition 4.1 (Delaunay Triangulation) *For given set P of points in dually flat space, a triangulation of P is* Delaunay triangulation $\mathrm{Del}(P)$ *if no points are included in the circumscribed sphere of any d-simplex in* $\mathrm{Del}(P)$.

However, some parts of the circumscribed sphere of the simplex of the Delaunay triangulation are not included: the parts correspond to faces removed from the ∇(or ∇^*)-Voronoi diagram. Such a structure may not be a triangulation since there are some deleted parts. Since the convex hull for given points is included in the space, their triangulation which is a subdivision of the convex hull is also included. Thus we define *Delaunay-type triangulation* in the dually flat space. That is, we define a triangulation based not on the circumscribed sphere but on the potential function. Not only the properties of Delaunay-type triangulation but the relations between this triangulation and the Voronoi diagram are described.

First we define ∇- and ∇^*-*Delaunay-type triangulations*. In the dually flat space there exist *potential functions* that can be used to construct Voronoi diagrams. We can also use potential functions to construct the triangulation. For given points consider a θ-coordinate system and suppose a potential function ψ. By adding one axis y to $[\theta]$, we can consider $[\theta, y]$ space. Then we can compute the value of potential function ψ at each point, i.e., lifting points $[\theta^{(i)}, \psi(\theta^{(i)})]$ and construct the ∇-convex hull for these points. When we project the lower envelope to the original space, the projection becomes a triangulation if any d points determine a hyperplane and any $k(> d)$ points are not on a hyperplane where d is the dimension of the space.

The above steps are summed up the following algorithm.

Algorithm 4.1 ($\nabla(\nabla^*)$-Delaunay-type Triangulation)
Input: a set of points in dually flat space;
Output: $\nabla(\nabla^*)$-Delaunay-type triangulation;

1. Lift up given points to potential function $\psi(\varphi)$.
2. Construct the $\nabla(\nabla^*)$-convex hull of these points in the $[\theta, y]([\eta, y])$-coordinate system.
3. Project the lower envelope of the convex hull to $\theta(\eta)$-coordinate system.

Here, we obtain the duality of ∇-Voronoi diagram and ∇^*-Delaunay-type triangulation. In Theorem 3.2 we consider that diagram (lower envelope of hyperplane arrangement) and the dual structure *convex hull*. There is a *duality* between the ∇-Voronoi diagram in the θ-coordinate system and the ∇^*-Delaunay-type triangulation in the η coordinate system. The duality can be understood as the correspondence between point and hyperplane in the dually flat space. The tangent hyperplane of ψ at p in the θ-coordinate system is expressed as

$$y = -\sum_{i=1}^{d} \theta^i \eta_i(p) + \varphi(p)$$

and is transformed to

$$(\eta_1, \eta_2, \cdots, \eta_d, -\varphi(p)).$$

Thus the hyperplane arrangement of given points in the coordinate system is transformed to the hyperplane arrangement whose hyperplane represents the *supporting hyperplane*, that is, the *facet* of the convex hull in the another coordinate. Moreover the projection of the upper envelope of the hyperplane arrangement is the $\nabla(\nabla^*)$-Voronoi diagram, and the projection of the lower envelope of the $\nabla^*(\nabla)$-convex hull is the $\nabla^*(\nabla)$-Delaunay-type triangulation. Therefore there also exists duality between the ∇-Voronoi diagram and the ∇^*-Delaunay-type triangulation.

Theorem 4.1 (Duality) *For given points in a dually flat space with a global coordinate system, there is the relations between $\nabla(\nabla^*)$-Voronoi diagram, without removing faces, and $\nabla^*(\nabla)$-Delaunay-type triangulation, respectively, called* duality *between these structures. The duality is given by transformation $\mathcal{D}$ from $[\eta, y]$ to $[\theta, y]$:*

$$\mathcal{D} : [\eta, -\varphi] \longleftrightarrow y = -\sum_{i=1}^{d} \theta^i \eta_i(p) + \varphi(p).$$

Moreover, $\nabla(\nabla^*)$-divergence is preserved by $\mathcal{D}$.

Proof: We prove the duality only of the ∇-Voronoi diagram and the ∇^*-Delaunay-type triangulation. Another correspondence that between the ∇^*-Voronoi diagram and the ∇-Delaunay-type triangulation is similarly shown. When removed faces may exist in the ∇-Voronoi diagram, we consider the whole projection of the upper envelope of hyperplane arrangement. We already explained that the structure is transformed to ∇^*-Delaunay-type triangulation.

So we state that the divergence is not changed by $\mathcal{D}$. Suppose a map from the tangent hyperplane of a potential function to the tangent point. This map is a bijection because the potential function is strictly convex. Since $\mathcal{D}$ is also a bijective from point to hyperplane, we can get the bijection from θ to η. Therefore, by the definition of divergence (Definition 2.7), the ∇-divergence is unchanged □

The duality in Euclidean space is easily understood. The two coordinate systems are the same and the two potential functions are also the same. Therefore the duality $\mathcal{D}$ is from $[\xi, y]$ to $[\xi, y]$ and the function from $[\xi]$ to $[\xi]$ is identity. Then the duality can be considered in the one coordinate system.

To describe the properties of Delaunay-type triangulation, we define some geometric structures in the dually flat space.

Definition 4.2 (∇-gravity Point, ∇-circumcenter, ∇-sphere, and ∇-circumscribed Sphere) *Suppose points $\{p^{(j)}; j = 1, \ldots, n\}$ in the d-dimensional dually flat space.*

∇-gravity point *A point p is a ∇-gravity point if p attains*

$$\min \sum_{i=1}^{n} D(p^{(i)} \| p). \tag{4.1}$$

∇-circumcenter *Suppose simultaneous equations*

$$D(p^{(i)} \| p) = D(p^{(j)} \| p) \ (\forall i, j = 1, \cdots, d+1) \tag{4.2}$$

for $d+1$ points in the dually flat space. The solution of these equations is called a ∇-circumcenter.

∇-sphere *For a point p in the dually flat space and a positive real number r, the set of point is a ∇-sphere if the ∇-divergence from p to the point of set is r.*

∇-circumscribed sphere *A ∇-sphere for a d-simplex in the d-dimensional dually flat space is a circumscribed sphere if all the vertices of the d-simplex are on the boundary of the sphere.*

The ∇^*-gravity point, the ∇^*-circumcenter, and the ∇^*-sphere are defined by substituting ∇^*-divergence $D^*(\cdot\|\cdot)$ for ∇-divergence $D(\cdot\|\cdot)$.

[**Remark**] These facts are shown immediately.

- A ∇-circumcenter does not always exist. If it exists, the point becomes the ∇^*-Voronoi point for $(d+1)$ points.
- A ∇-sphere is always included in the space.

The following lemma gives an easy way to calculate the ∇-gravity points.

Lemma 4.1 *Suppose points $\{p^{(j)}; j = 1, \ldots, n\}$ in the d-dimensional dually flat space with a global coordinate system. The ∇-gravity point p in the θ-coordinate system can be uniquely determined to be*

$$\theta^j(p) = \frac{1}{n}\sum_{i=1}^{n} \theta^j(p^{(i)}) \tag{4.3}$$

where $\theta^j(p^{(i)})$ is the value of the jth coordinate of $p^{(i)}$.

Proof: Differentiate by θ the criterion (4.1)

$$\frac{\partial}{\partial \eta_j}\sum_{i=1}^{n} D(p^{(i)}\|p) = n \cdot \theta^j(p) - \sum_{i=1}^{n} \theta^j(p^{(i)}) \quad (j = 1, \cdots, d). \tag{4.4}$$

The quadratic differential is positive definite because that is metric (2.8). If there are no solutions to the above equations, the point is a ∇-gravity point. The vanishing point is given by equation (4.3). □

[**Remark**] ∇^*-gravity points are calculated similarly. Thus the point is also expressed by

$$\eta_j(p) = \frac{1}{n}\sum_{i=1}^{n} \eta_j(p^{(i)}). \tag{4.5}$$

In addition, equation (4.5) is the same as equation (2.4). If this lemma for ∇^*-gravity point is limited to the statistical parametric space of exponential family, then the above equation expresses the maximum likelihood estimator for n points (Section 2.1.2).

Moreover, some graphs are defined for a set of points in d-dimensional Euclidean space. So that the set is contained in Euclidean space, suppose that these graphs are embedded in $\mathbb{R}^d$. For example,

Gabriel Graph (GG) A graph whose vertices are points in $\mathbb{R}^d$, with edge (x, y) iff points x and y defined the diameter of an empty sphere.

Relative Neighborhood Graph (RNG) A graph whose vertices are points in $\mathbb{R}^d$, with edge (x, y) iff there exists no points z such that z is closer to x than y is and z is closer to y than x.

There are for these graphs the following relations [11]:

$$\text{EMST} \subseteq \text{RNG} \subseteq \text{GG} \subseteq \text{Del}.$$

EMST is the *Euclidean minimum spanning tree*, which is the minimum spanning tree and whose weight is the Euclidean distance between two end points, and Del is the *Delaunay triangulation* of given points.

Since in the dually flat space the divergence does not satisfy the symmetry of distance, all graphs are not dealt with. That is, GG can be considered, but RNG, and EMST cannot. Thus the GG in the dually flat space is defined.

Definition 4.3 ($\nabla(\nabla^*)$-Gabriel Graph ($\nabla(\nabla^*)$-GG)) *Suppose a set P of points in the dually flat space. A graph is a* ∇-Gabriel graph *if whose vertex is P and edge (x, y) iff ∇-sphere, whose center is ∇-gravity point c of x, y and radius is ∇-divergence from x to c, is empty, that is, no points is contained in the interior of the sphere. The* ∇^*-Gabriel graph *is defined by replacing ∇ with ∇^*.*

Delaunay-type triangulation satisfies the following lemma.

Theorem 4.2 *For any given points in the dually flat space, the ∇-Delaunay-type triangulation includes the ∇-Gabriel graph. That is,*

$$\nabla\text{-GG} \subseteq \nabla\text{-Del}.$$

∇-Del *is the ∇-Delaunay-type triangulation. There is a similar relation for the ∇^*-Gabriel graph and the ∇^*-Delaunay-type triangulation.*

Proof: Let (x, y) be a geodesic segment that connects x and y and that becomes an edge of the ∇-Gabriel graph. In Theorem 4.1 we proved the duality of the ∇-Delaunay-type triangulation and the ∇^*-Voronoi diagram without removed faces. Compare the definition of the ∇-GG and of the ∇^*-Voronoi diagram. The set of equi-divergence points to two generators includes the ∇-gravity point of these points, and the ∇-gravity point lies on the ∇-geodesic segment joining generators (see Fig.3). That is, because the ∇-gravity point is given by the average of two generators in the θ-coordinate system (Lemma 4.1). Thus this geodesic segment is equal to the edge of graph.

Finally we show the inclusion for any edge of ∇-Gabriel graph. The existence condition of the edge is that there are no points in the sphere whose center is ∇-gravity and whose radius is the divergence. Since there is no points in the sphere, the ∇^*-Voronoi facet exists. That is, there isa the geodesic segment connecting the two generators. □

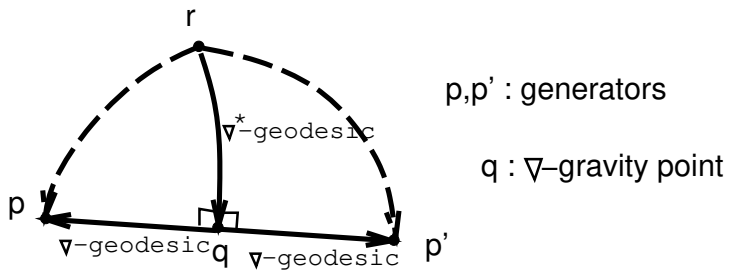

Figure 3: Two generators and the Pythagorean theorem in the θ-coordinate system

Here we prove that Delaunay-type triangulation is the "best" of all possible triangulations in the dually flat space. That is, we prove that the ∇-Delaunay-type triangulation minimizes the maximum radius of the *minimum enclosing ∇-sphere* by ∇-divergence.

First, we define the *minimum enclosing ∇-sphere*.

Definition 4.4 (Minimum Enclosing ∇-Sphere) *Suppose $d+1$ points in the d-dimensional dually flat space. A ∇-sphere is the* minimum enclosing ∇-sphere *if the sphere includes $d+1$ points and the radius is minimized by ∇-divergence from center point.*

Lemma 4.2 *For given points $P = \{p^{(j)}; j = 1, \ldots, d+1\}$ in the d-dimensional dually flat space, let X be a linear combination of given points. That is, $X = \sum_{j=1}^{d+1} \lambda_i p^{(i)}$ in the θ-coordinate system where $\lambda_i > 0, \sum_i \lambda_i = 1$. The following equation is obtained:*

$$F(X) := \sum_{i=1}^{d+1} \lambda_i D(p^{(i)} \| X) = R - D(X_C \| X) \tag{4.6}$$

where X_C is the ∇-circumcenter for P and $R = D(p^{(j)} \| X_C)$ is constant $(j = 1, \ldots, d+1)$ when X_C exists.

Proof:

$$
\begin{aligned}
R - D(X\|X_C) &= \sum_{i=1}^{d+1} \{R - D(X\|X_C)\} = \sum_{i=1}^{d+1} \lambda_i \left[D(p^{(i)}\|X_C) - D(X\|X_C)\right] \\
&= \sum_{i=1}^{d+1} \lambda_i \left[\psi(p^{(i)}) + \varphi(X_C) - \sum_j \theta^j(p^{(i)})\eta_j(X_C) - \psi(X) - \varphi(X_C) + \sum_j \theta^j(X)\eta_j(X_C)\right] \\
&= \sum_{i=1}^{d+1} \lambda_i \left[D(p^{(i)}\|X) + \sum_j \left\{\theta^j(p^{(i)}) - \theta^j(X)\right\}\{\eta_j(X) - \eta_j(X_C)\}\right] \\
&= \sum_{i=1}^{d+1} \lambda_i D(p^{(i)}\|X) + \sum_j \left\{\theta^j(X) - \theta^j(X)\right\}\{\eta_j(X) - \eta_j(X_C)\} \\
&= \sum_{i=1}^{d+1} \lambda_i D(p^{(i)}\|X) = F(X)
\end{aligned}
$$

□

A proof of the above equation restricted to statistical parametric space, above equation for Kullback-Leibler divergence, is already shown in [12]. Here we prove it in general for a dually flat space with a global coordinate system.

Moreover, by the above equation we can get the maximized point of $F(X)$.

Lemma 4.3 *In Lemma 4.2, if the ∇-circumcenter of the $d+1$ points is included in the d-simplex, $F(X)$ is maximized at the ∇-circumcenter. Otherwise the function is maximized at the boundary of the d-simplex.*

Proof: By Lemma 4.2

$$F(X) = R - D(X_C\|X).$$

Thus the $F(X)$ attains to maximum R at $X = X_C$ if X_C contained in the simplex. If X_C is not included in the simplex, the function becomes minimum at a point of the boundary of the simplex. Therefore suppose a enclosing ∇-sphere, F is maximized at the ∇-circumcenter among the point set $P' \subset P$ on the ∇-sphere as the e-dimensional space where $e+1$ is the number of P'.

In addition, if X_C does not exist in the space, then we consider ∇-circumscribed sphere of the face of the simplex, which contains all vertices of the simplex. The sphere with the smallest radius among such spheres is minimum enclosing sphere for the simplex. In this case the center of the sphere is on the boundary of the simplex. Because $F(X)$ maximize at the center of such sphere for the subset P' of points and λ_i for $P \setminus P'$ equal to 0. □

In addition, we define some functions. For $P = \{p^{(j)}; j = 1, \ldots, n\}$ in d-dimensional dually flat space the function $f(X)$ is defined at every point X in the ∇-convex hull ∇-CH(P):

$$\lambda_i \geq 0, \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \sum \lambda_i p^{(i)} = X, \tag{4.7}$$

$$F(X, \lambda) = \sum_{i=1}^{n} \lambda_i D(X\|p^{(j)}), \tag{4.8}$$

$$f(X) = \min_{\lambda} F(X, \lambda). \tag{4.9}$$

Theorem 4.3 *For $\min_\lambda F(X, \lambda)$ of a fixed point X, the non-zero values of λ_i are determined uniquely by the vertices of the Delaunay simplex, which is the simplex of ∇-Delaunay-type triangulation, containing the point X. Thus $f(X)$ is given by Lemma 4.3:*

$$f(X) = f_t(X) = R_t - D(X_{C_t}\|X)$$

where X_{C_t} and R_t are respectively the center and the radius of the minimum enclosing ∇-sphere of the simplex.

Proof: The ∇-Delaunay-type triangulation can be provided by the potential function ψ and the projection of the lower hull of ∇-convex hull (Algorithm 4.1).

Now we consider a point

$$\left(\sum_{i=1}^{n} \lambda_i p^{(i)}, \sum_{i=1}^{n} \lambda \psi(p^{(i)})\right) = (X, F(X,\lambda) + \psi(X)),$$

under the condition (4.7). This point represents all the points of ∇-CH(P). Moreover the minimum of F for X is given by the point with the lowest y-coordinate; therefore, the point lies on the lower-hull of the convex hull. This simplex is included in the ∇-Delaunay-type triangulation containing X. □

Finally we get the theorem about the minimum enclosing ∇-sphere.

Theorem 4.4 *The maximum radius of the minimum enclosing ∇-sphere of the simplex of the ∇-Delaunay-type triangulation, which is the ∇-divergence from the center to the points on the sphere, is less than the maximum radius of the minimum enclosing ∇-sphere of any other triangulation of the set of points.*

Proof: A triangulation $\mathcal{T}$ for a set P of points is considered. The function $F_{\mathcal{T}}(X)$ is defined at each point X in ∇-CH(P) and is the same as the equation (4.6) for the simplex $T \in \mathcal{T}$ that contains the point X. Let $F_{\mathcal{D}}$ be a corresponding function for ∇-Delaunay-type triangulation $\mathcal{D}$. Let X_T and X_D be the points in ∇-CH(P) where $F_{\mathcal{T}}(X)$ and $F_{\mathcal{D}}(X)$ attain their maxima. Moreover, let R_T and R_D be the respectively ∇-divergences from X_T and X_D to the vertex of the simplex.

By Lemma 4.3 and Theorem 4.3 the following relation is proved :

$$R_T = F_{\mathcal{T}}(X_T) \geq F_{\mathcal{T}}(X_D) \geq F_{\mathcal{D}}(X_D) = R_D.$$

Because the first inequality is based on the maxima of X_T for $\mathcal{T}$ and the second is by Theorem 4.3. Therefore we can get the result. □

We proved Theorem 4.4 for ∇-Delaunay-type triangulation, and there is a similar theorem for ∇^*-Delaunay-type triangulation.

Theorem 4.5 *The maximum radius of the minimum enclosing ∇^*-sphere of ∇^*-Delaunay-type triangulation, which is the ∇^*-divergence from the ∇^*-center to the points on the sphere, is less than the maximum radius of the minimum enclosing ∇^*-sphere of any other triangulation of the point set.*

Proof: The above lemmas of the ∇^* version can be proved by exchanging the role of ∇ and ∇^*. Those lemmas can be used to prove this theorem. □

[**Remark**] Theorem 4.4 includes the Euclidean case. This is proved in [26].

In this section we propose the *Delaunay-type triangulation* and prove some properties. Since this theory is a natural extension of Euclidean computational geometry, that triangulation is also characterized by some other theorems by potential function.

5 Conclusion

This paper described two new kinds of computable geometric structures, the *Voronoi diagram* and the *Delaunay-type triangulation*, in a *dually flat space* with a global coordinate system. These structures have the relation with duality and are efficiently computed by potential functions as same as the case of Euclidean. To explain more details:

Voronoi diagram Lifting up given points to the potential function and suppose a tangent hyperplane at each point. The lowest part of such a hyperplane arrangement projects on the original space.

Delaunay-type triangulation Lifting up given points to the potential function and compute a convex hull. The lowest part of such a convex hull (i.e., the lower hull) projects on the original space.

These algorithms for computation are based on the equality of the divergence and the difference between the potential function and the tangent hyperplane. In the case of Euclidean space, two potential functions φ, ψ coincide and two coordinate systems $[\theta], [\eta]$ are also identified. The higher-order Voronoi diagram is dealt with by using that hyperplane arrangement. Moreover, it is proved to minimize the maximum minimum enclosing sphere for Delaunay-type triangulation among any triangulation.

Dually flat space contains *statistical parametric space* [1], the *feasible region of linear programming* [30], the *parametric family of invertible linear system* [2], etc. Since such spaces exist, it is important to investigate geometric structure for understanding spaces. Specifically, the Voronoi diagram in statistical parametric space can be applied to *clustering* by divergence because of the relation with maximum likelihood estimation. Consequently, this research not only provides a new aspect of computational geometry but also has implications for information theory, linear programming, etc.

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