Aperture-Angle Optimization Problems in 3 Dimensions*

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1 Introduction

In this paper we consider aperture angle optimization problems in 3 dimensions. Such problems have a long history dating at least as far back as the mathematician Regiomontanus (Johannes Muller) [2]. In 1471 Regiomontanus posed the following problem (now known as Regiomontanus' Maximum Problem). At what point of the earth's surface does a perpendicularly suspended rod appear longest? In other words, at what point is the aperture angle at a maximum? A solution of this problem was published by Ad. Lorsch in vol. XXIII of *Zeitschrift fur Mathematik und Physik*, assuming the earth is flat.

Here we consider generalizations of Regiomontanus' maximum problem and the problems considered in [1] to three dimensions. In particular, let P be a convex polyhedron in $\mathbf{R}^3$. Let $[a,b]$ be a line segment with end points a, b and ν a point at which a viewer is located, also in $\mathbf{R}^3$. The aperture angle of $[a,b]$ from point ν, denoted by $\theta(\nu)$, is the interior angle at ν of the triangle $\Delta(a,b,\nu)$. Given a polyhedron P not intersecting a given segment $[a,b]$ we consider the problem of computing $\theta_{max}(\nu)$ and $\theta_{min}(\nu)$, the maximum and minimum values of $\theta(\nu)$ as ν varies over all points in P. We obtain two characterizations of $\theta_{max}(\nu)$. Along the way we solve several interesting special cases of the above problems and establish linear upper and lower bounds on their complexity under several models of computation.

2 Viewing from a plane

Let the length of $[a,b]$ be l and denote the distance between ν and the end-points of $[a,b]$ by l_a and l_b, respectively. Then, by the law of cosines the aperture angle is given by:

$$\theta = \arccos\left(\frac{{l_a}^2 + {l_b}^2 - l^2}{2 l_a l_b}\right)$$

In this planar-constraint version of the problem we are interested in determining the maximum aperture angle $\theta_{max}(\nu)$ where ν varies over the entire xy-plane.

In general the function looks like a "volcano" with a tilted "crater" that reduces down to a point where the line through $[a,b]$ intersects the xy-plane. Since this function contains a singularity it is difficult to obtain approximate solutions with constrained numerical optimization methods since these techniques require the function to be twice differentiable [3]. Fortunately as in the two dimensional version of this problem [1] we can obtain a simple characterization of the solution that allows us to compute it efficiently.

3 Viewing from a line

Let $[a,b]$ be a line segment and L be a line not intersecting $[a,b]$ in $\mathbf{R}^3$ on which the viewer is constrained to patrol. In this constrained version of the problem we are interested in determining the maximum aperture angle $\theta_{max}(\nu)$ where ν varies over the entire line.

We now describe our first characterization of the solution for $\theta_{max}(\nu)$ where c denotes the center of the aperture circle.

Lemma 3.1 *$\theta_{max}(\nu)$ is determined by the point ν^* in L that yields the smallest aperture circle with the property that segment $[c,\nu^*]$ is orthogonal to L.*

The characterization of the solution provided by lemma 3.1 allows us to compute $\theta_{max}(\nu)$ in constant time if the basic real RAM is extended to allow the computation of cube roots and trigonometric functions as primitives. Three candidate aperture circles can be obtained from the roots of a cubic equation

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which can be derived from the constraints of the problem imposed by the characterization. Expressions for computing these roots may be obtained from Cardano's formulas and, depending on the configuration of $[a,b]$ and L, can sometimes be represented as algebraic functions of nested radicals and in some cases trigonometric functions [4].

We now describe our second characterization of the solution for $\theta_{max}(\nu)$. The first characterization presented above involves *four* constraints that must be satisfied by the aperture circle. The second characterization involves minimizing the radius of a sphere subject to *three* constraints. This characterization has no computational advantage over the previous one because it leads to the same cubic equation. However, together, the two characterizations provide a deeper geometric understanding of the problem.

Lemma 3.2 *$\theta_{max}(\nu)$ is determined by the point ν^* in L that is the point of tangency of L with the smallest sphere S containing a and b on its boundary.*

4 Viewing from a polyhedron

Let P be a convex polyhedron in $\mathbf{R}^3$ not intersecting the line segment $[a,b]$. We are interested in the problems of determining $\theta_{max}(\nu)$ and $\theta_{min}(\nu)$ where ν varies over the entire polyhedron P. We assume that the polyhedron is represented in the form of a *doubly-connected-edge-list* (DCEL).

4.1 Computing $\theta_{max}(\nu)$

First we show that only the boundary of P (denoted by $bd(P)$) need be searched.

Lemma 4.1 *$\theta_{max}(\nu)$ is determined by $bd(P)$.*

In fact we may restrict further our search for $\theta_{max}(\nu)$ to a smaller region of the boundary of P "facing" $[a,b]$, a region we refer to as the inner facets of P with respect to segment $[a,b]$. We say that two points x and y in $\mathbf{R}^3$ are *visible* (or *see* each other) if the interior of $[x,y]$ does not intersect the interior of P. We say that a point x sees a set S if x sees every element of S. We relate the solution region to the subset of $bd(P)$ that has the property that every point of this subset sees $[a,b]$.

Lemma 4.2 *$\theta_{max}(\nu)$ is determined by a point ν^* that sees $[a,b]$.*

Theorem 4.3 *Given a line segment $[a,b]$ and a disjoint convex n-vertex polyhedron P, $\theta_{max}(\nu)$ may be computed in optimal $\Theta(n)$ time.*

4.2 Computing $\theta_{min}(\nu)$

We assume that the line $L(a,b)$ through the segment $[a,b]$ does not intersect polyhedron P. In the previous problem $\theta_{max}(\nu)$ may be realized by a point in the interior of some facet. On the other hand the $\theta_{min}(\nu)$ problem is fundamentally different in the sense that the solution must lie on the boundary of a facet of P. Furthermore, the solution need not lie on a vertex of P, as it necessarily must in the 2-dimensional version of this problem. In 3-dimensions the solution may lie in the interior of an edge of P, thus necessitating the results on viewing from a line.

Lemma 4.4 *$\theta_{min}(\nu)$ is determined by an edge of P.*

We show that $\theta_{min}(\nu)$ is determined by an edge of the "outer facets" of P with respect to segment $[a,b]$. Let $B(P)$ denote the portion of $bd(P)$ on $CH(P \cup [a,b])$, where CH denotes the convex hull. We call the facets of $B(P)$ the outer facets of P with respect to $[a,b]$.

Lemma 4.5 *$\theta_{min}(\nu)$ is determined by $B(P)$.*

Lemma 4.6 *$\theta(\nu)$ may have $\Omega(n)$ local minima.*

Theorem 4.7 *Given a line segment $[a,b]$ and a disjoint convex n-vertex polyhedron P, $\theta_{min}(\nu)$ may be computed in optimal $\Theta(n)$ time.*

References

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