Subexponential-time algorithms for minimum weight triangulation and related problems *

Andrzej Lingas
Department of Computer Science, Lund University
Box 118, S-22100 Lund, Sweden
E-mail: Andrzej.Lingas@dna.lth.se

Extended abstract

We consider the problem of computing optimal straight-line subgraphs of the complete Euclidean graph on a planar n-point set S. The subgraphs are typically required to fall into such natural classes of planar straight-line graphs as (diagonal) triangulations of S or k-gon diagonal partitions of S. The optimality criteria range from the elementary ones as the minimum total edge length to more complicated ones that optimize the minimum or maximum of the values of some specific parameter for the resulting k-gons (e.g., minmax diameter).

We observe that similarly as the traveling salesman problem in the plane [2], many of the problems can be exactly solved in subexponential time, more precisely, in time $2^{O(\sqrt{n}\log n)}$, by applying Miller's simple cycle planar separator theorem [1].

We shall shortly illustrate our observation by considering the famous problem of computing the minimum weight triangulation of S. Let T be a minimum weight triangulation of S. By Miller's theorem, there is a simple cycle in T on $O(\sqrt{n})$ points which separates S into two parts, each containing at most $\frac{2}{3}n$ points. Now, the general idea is to "guess" such a cycle and recursively find the minimum weight triangulations of the two resulting figures (one of the them is a simple polygon with point holes, the other consists of a single polygonal hole and a subset of S). Instead of guessing such a separating cycle, we enumerate balanced simple separator cycles by considering all permutations of $O(\sqrt{n})$-element subsets of S and testing whether they yield simple cycles separating S in the required balanced way. It takes time $2^{O(\sqrt{n}\log n)}$ as the number of the subsets and their permutations is $2^{O(\sqrt{n}\log n)}$ and the tests can be done easily in polynomial time. Next, we run recursively our procedure for all of the enumerated cycles. Among the pairs of optimal solutions for the two subproblems induced by each of the cycles, we choose that yielding the shortest triangulation of S.

The recursion depth is $O(\log n)$. At its bottom, we have finite subproblems that can be solved in constant time. Generally, a subproblem can be described by $O(\log n)$ fragments of bounding cycles of total length $O\left(\sum_{i=1}\sqrt{(\frac{2}{3})^i n}\right) = O(\sqrt{n})$. Hence, the total number of subproblems is $\binom{n}{O(\sqrt{n})}O(\sqrt{n})!\binom{O(\sqrt{n})}{O(\log n)}$, i.e., $2^{O(\sqrt{n}\log n)}$ and a single subproblem can be solved on the basis of the smaller ones in time $2^{O(\sqrt{n}\log n)}$.

References

[1] G. Miller. *Finding small simple cycle separators for 2-connected planar graphs.* Proc. 16th Ann. ACM Symp. on Theory of Computing (1984) pp. 376-382.

[2] M. Bern and D. Eppstein. Approximation Algorithms for Geometric Problems. In D. S. Hochbaum, editor, *Approximation Algorithms for $\mathcal{NP}$-Hard Problems*, chapter 8, pages 296–339. PWS Publishing Company, Boston, MA, 1996.

*Research supported in part by TFR grant 96-278.