Straight-line embeddings of three rooted trees in the plane

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We consider finite planar graphs without loops or multiple edges. Let G be a planar graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $|G|$ the order of G, that is, $|G| = |V(G)|$. Given a planar graph G, let P be a set of $|G|$ points in the plane (2-dimentional Euclidean space) in general position (i.e., no three of them are collinear). Then G is said to be *line embedded onto* P or *stright-line embedded onto* P if G can be embedded in the plane so that every vertex of G corresponds to a point of P, every edge corresponds to a straight-line segment, and no two straight-line segments intersect except their common end-point. Namely, G is line embedded onto P if there exists a bijection $\phi : V(G) \to P$ such that two points $\phi(x)$ and $\phi(y)$ are joined by a straight-line segment if and only if x and y are joined by an edge of G and all two distinct open straight-line segments have no point in common. We call such a bijection a *line embedding* or a *straight-line embedding* of G onto P.

In this paper we consider a line embedding having one more property. Let G be a planar graph with n specified viertices $v_1, v_2, \ldots, v_n$, and P a set of $|G|$ points in the plane in general position containing n specified points $p_1, p_2, \ldots, p_n$. Then we say that G is *strongly line embedded onto* P if G can be line embedded onto P so that for every $1 \leq i \leq n$, v_i corrresponds to p_i, that is, if there exists a line embedding $\phi : V(G) \to P$ such that $\phi(v_i) = p_i$ for all $1 \leq i \leq n$. The line embedding mentioned above is called a *strong line embedding* of G onto P. A tree with one specified vertex v is usually called a *rooted tree* with root v. Given n disjoint rooted trees T_i with root v_i, $1 \leq i \leq n$, the union $T_1 \cup T_2 \cup \cdots \cup T_n$, whose vertex set is $V(T_1) \cup V(T_2) \cup \cdots \cup V(T_n)$ and whose edge set is $E(T_1) \cup E(T_2) \cup \cdots \cup E(T_n)$, is called a *rooted forest* with roots $v_1, v_2, \ldots, v_n$, which are specified vertices of it.

We begin with the following theorem, which was conjectured by Perles [5] and partially solved by Pach and Törőcsik [4]; a simpler proof can be found in Tokunaga [7]. Another related result can be found in [1].

Theorem A ([2]) *A rooted tree T can be strongly line embedded onto every set of $|T|$ points in the plane in general position containing a specified point.*

Kaneko and Kano [3] extended the problem as follows:

Theorem B ([3]) *A rooted forest F consisting of two rooted trees can be strongly line embedded onto every set of $|F|$ points in the plane in general position containing two specified points.*

It is easy to see that there exist rooted forests consisting of four rooted trees which cannot be strongly line embedded onto certain sets of points in the plane in general position containing four specified points. Thus, 'three' is the the bound. In [3], the authors conjectured that 'two' in Theorem B can be incresed to 'three'. In this paper we show the counterexample of this conjecture. We then introduce the definition of *semistrong line enbedding* by weakening the condition of strong line enbedding. G is said to be *semistrongly line embedded onto P* if there exists a line embedding $\phi : V(G) \to P$ such that $\phi(\{v_1, v_2, \cdots, v_n\}) = \{p_1, p_2, \cdots, p_n\}$. In this paper, we prove the following theorem:

Theorem 1 *A rooted forest F consisting of three rooted trees can be semistrongly line embedded onto every set of $|F|$ points in the plane in general position containing three specified points.*

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