Balanced partitions of two sets of points in the plane

Atsushi Kaneko
Kogakuin University, Shinjuku-ku, Tokyo 163-8677 Japan
e-mail: kaneko@ee.kogakuin.ac.jp
M. Kano
Ibaraki University, Hitachi 316-8511 Japan
e-mail: kano@cis.ibaraki.ac.jp

For a set P of points in the plane, we denote by conv(P) the *convex hull* of P, which is the smallest convex set containing P. In [1], we proved the following theorem.

Theorem A *Let m be a positive integer, and let S_1, S_2 and T be three disjoint sets of points in the plane such that no three points of $S_1 \cup S_2 \cup T$ are on the same line (i.e., no three points of it are collinear) and $|T| = (m-1)|S_1| + m|S_2|$. Put $q = |S_1 \cup S_2|$. Then $S_1 \cup S_2 \cup T$ can be partitioned into q subsets $P_1, P_2, \cdots, P_q$ which satisfy the following three conditions: (i) $\mathrm{conv}\,(P_i) \cap \mathrm{conv}\ (P_j) = \phi$ for all $1 \le i < j \le q$; (ii) $|P_i \cap (S_1 \cup S_2)| = 1$ for all $1 \le i \le q$; and (iii) $|P_i \cap T| = m-1$ if $|P_i \cap S_1| = 1$, and $|P_i \cap T| = m$ if $|P_i \cap S_2| = 1$.*

In view of theorem A with $S_1 = \emptyset$, we gave the following conjecture in [1].

Conjecture B *Let $m \ge 2$, $n \ge 2$ and q be positive integers. Let S and T be two disjoint sets of points in the plane such that no three points of $S \cup T$ are collinear, $|S| = nq$ and $|T| = mq$. Then $S \cup T$ can be partitioned into q subsets $P_1, P_2, \cdots, P_q$ satsifying the following two conditions: (i) $\mathrm{conv}\,(P_i) \cap \mathrm{conv}\,(P_j) = \phi$ for all $1 \le i < j \le q$; and (ii) $|P_i \cap S| = n$ and $|P_i \cap T| = m$ for all $1 \le i \le q$.*

The above conjecture is true when $q = 2$ because the conjecture with $q = 2$ is equivalent to well-known discrete Ham Sandwich Theorem on the plane ([2] p.212). In this paper we show that the conjecture is ture in the case of $n = 2$.

Theorem 1 *Let $m \ge 2$ and $q \ge 1$ be integers and let S and T be two disjoint sets of points in the plane such that no three points of $S \cup T$ are collinear, $|S| = 2q$ and $|T| = mq$. Then $S \cup T$ can be partitioned into q disjoint subsets $P_1, P_2, \cdots, P_q$ satisfying the following two conditions:*

(i) $\mathrm{conv}\,(P_i) \cap \mathrm{conv}\,(P_j) = \phi$ for all $1 \le i < j \le q$; and

(ii) $|P_i \cap S| = 2$ and $|P_i \cap T| = m$ for all $1 \le i \le q$.

From the proof of this theorem, we can obtain $O(n^2 log n)$ time algorithm for finding the required partition, where $n = |S \cup T|$.

In order to prove this theorem, we need the following lemma and claims.

Lemma 2 *Let S and T be two disjoint sets of points in the plane given in Theorem 1. If there exist two lines l_1 and l_2 such that $|R(l_1) \cap S| = |R(l_2) \cap S|$ and $|R(l_1) \cap T| < |R(l_2) \cap T|$, then for every integer i, $|R(l_1) \cap T| \leq i \leq |R(l_2) \cap T|$, there exists a line l_3 such that $|R(l_3) \cap S| = |R(l_1) \cap S|$ and $|R(l_3) \cap T| = i$.*

Claim 3 *For every integer i, $0 \leq i \leq q - 1$, there exist a line l that passes through two distinct points of S and satisfies $|R(l) \cap S| = i$.*

Claim 4 *We may assume that q is odd.*

Claim 5 *we may assume that for every line l such that $2 \leq |R(l) \cap S| = 2j \leq q - 1$, we have $|R(l) \cap T| > mj$.*

References

[1] A. Kaneko and M. Kano, A balanced partition of points in the plane and tree embedding problems, submitted.

[2] J.E.Goodman and J.O'Rourke, *Handbook of Discrete and Compuational Geometry*, CRC Press, (1997)

Speaker: M. Kano
Department of Computer and Information Sciences
Ibaraki Univeristy, Hitachi, 316-8511
Japan
e-mail:kano@cis.ibaraki.ac.jp
fax:+81-294-37-1429