On a disjoint quadrilateral partition for a given point set

KIYOSHI HOSONO*

and

MASATSUGU URABE†

Department of Mathematics, Tokai University,
3-20-1 Orido, Shimizu, Shizuoka, 424-8610 Japan

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Let P be a set of n points in the plane, no three collinear. A partition of P is called a *convex partition* if P is partitioned by t subsets $P_1, P_2, \ldots, P_t$; $\sum_{i=1}^{t} |P_i| = n$, such that each $CH(P_i)$ is an $|P_i|$-gon for any i. Then we say that the convex partition has *size* t. A convex partition of P is called a *disjoint partition*, denoted by $\Pi(P)$, if each $CH(P_i)$ is disjoint from the others; $CH(P_i) \cap CH(P_j) = \emptyset$ for any pair of i, j. Let $f(P)$ denote the minimum size of any disjoint partition of $\Pi(P)$. Define $F(n) = max\{f(P)\}$, over all sets P of n points in the plane. It is showed that $\lceil (n-1)/4 \rceil \leq F(n) \leq \lceil 2n/7 \rceil$ in [7]. Now we improve on the upper bounds [5]:

Theorem 1

$$F(n) \leq \left\lceil \frac{5}{18}\, n \right\rceil \quad for \quad n \geq 1.$$

Theorem 2

$$F(n) \leq \frac{3n+1}{11} \quad for \quad n = 11 \cdot 2^{k-1} - 4 \quad (k \geq 1).$$

We conjecture that $F(n) \cong n/4$, that is, it means that P can be partitioned into all the convex quadrilaterals on the average. Next we consider the maximum number of disjoint convex quadrilaterals [6].

Given a point set P, we denote the maximum number of convex k-gons by $s_k(P)$ in any disjoint partition $\Pi(P)$. Moreover we define $S_k(n) = min\{s_k(P)\}$, over all sets P of n points in the plane. For a given n points set, it is trivial that $S_k(n)$ is a decreasing function for k. We can easily show that the following results by [1],[2],[3] and [4].

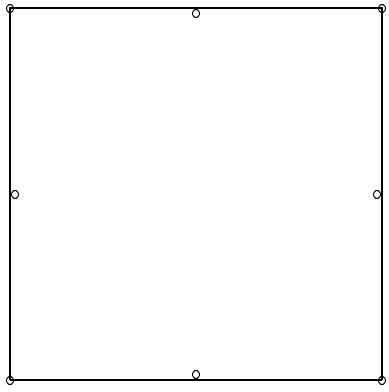
Theorem 3

(i) $S_1(n) = n$ (ii) $S_2(n) = \lfloor n/2 \rfloor$
(iii) $S_3(n) = \lfloor n/3 \rfloor$ (iv) $S_4(n) \geq \lfloor n/5 \rfloor$
(v) $S_5(n) \geq \lfloor n/10 \rfloor$
(vi) $S_k(n) = 0 \quad for \quad k \geq 7.$

Our interesting problem is to determine the exact values for $S_k(n)$. In particular, we investigate the values for $S_4(n)$. It is trivial that $S_4(n) = 0$ for $1 \leq n \leq 4$. In Theorem 3, we can easily show $S_4(5) = 1$.

* *e-mail address*: hosono@scc.u-tokai.ac.jp
† *e-mail address*: qzg00130@scc.u-tokai.ac.jp

What is the maximum number of n satistying $S_4(n) = 1$? There is the following eight points set P satisfying $s_4(P) = 1$. Therefore $S_4(8) = 1$ holds.



On the other hand, for every nine points set, we can always find two disjoint convex quadrilaterals.

Lemma 4 $S_4(9) = 2$.

Now we introduce the useful construction, namely Division Lemma.

Lemma 5 (Division Lemma) For any set of $2m + 4$ points in the plane, no three collinear, we can divide the plane into three disjoint convex regions such that one contains a convex quadrilateral and others contain m points each.

The following Theorems are constructed by Division Lemma.

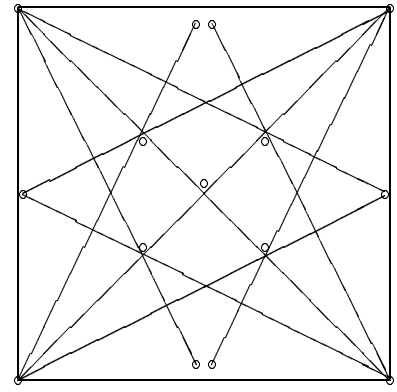
Theorem 6

$$S_4(n) \geq \left\lfloor \frac{5}{22} n \right\rfloor .$$

Theorem 7

$$S_4(n) \geq \left\lfloor \frac{3n-1}{13} \right\rfloor \ for\ n = 13 \cdot 2^{k-1} - 4 \ (k \geq 1).$$

In the case of $S_5(n)$, we find the following fifteen points set P with $s_5(P) = 1$. That is, $S_5(15) = 1$ holds.



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