Geometric tree graphs of points in the plane

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1 Introduction

Given a set P of points in the plane, let $\mathcal{T}(P)$ be the set of non-crossing spanning trees of P (edges are straight line segments and do not cross). In this paper we study the *geometric tree graph* $T(P)$, defined as the graph having $\mathcal{T}(P)$ as vertices and where two non-crossing spanning trees T_1 and T_2 are adjacent if $T_2 = T_1 - e + f$, for some edges e and f (Fig. 1). The combinatorial version of the problem has been largely studied (see for instance [4]).

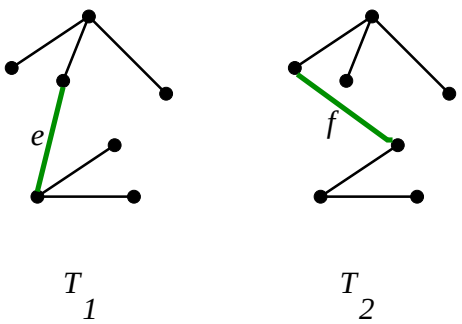

Figure 1: Two trees adjacent in $T(P)$

Avis and Fukuda [1] showed that $T(P)$ is connected for every set of points P in general position, and that the diameter of $T(P)$ is at most $2n - 4$ if $n = |P|$. García et al. [2] showed that $|\mathcal{T}(P)|$ is minimum when P is in convex position.

In Section 2 we summarize the results for points in convex position obtained in [3]. Section 3 contains a summary of results from ongoing research in the case of points in general position.

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2 Points in convex position: the graph G_n

As two sets of points in convex position are equivalent with respect to their non-crossing spanning trees, every set of n points in convex position has the same geometric tree graph, denoted by G_n (Fig. 2).

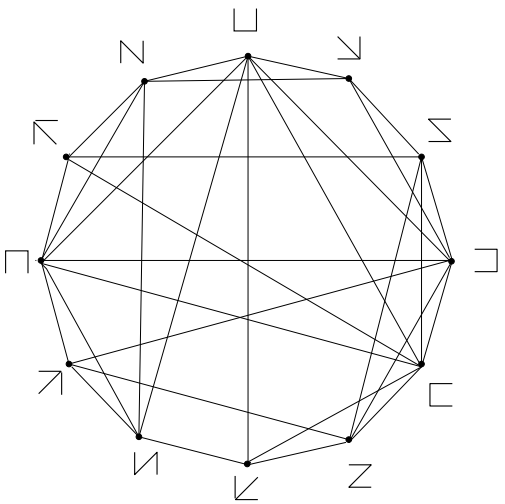

Figure 2: The graph G_4.

Next follows a list of properties satisfied by G_n (see [3]), showing that this is a combinatorial object interesting in itself. We recall that the *eccentricity* $e(T)$ of a tree $T \in T(P_n)$ is the maximum distance of T to any other tree in $T(P_n)$. The *radius* of a graph is the minimum of the eccentricities.

Theorem 2.1 *The minimum degree of G_n is 2n-4, and the stars are the only trees where this degree is reached.*

Theorem 2.2 *The maximum degree of G_n is $\binom{n+1}{3} - n + 1$ and only the spanning paths (which*

we call chains) which consist of all the edges of the convex hull but one, reach this degree.

Theorem 2.3 *The radius of G_n is equal to $n-2$ and the center is formed by the n stars and the n chains.*

Theorem 2.4 *Let D_n be the diameter of G_n. Then $D_n \geq 3n/2 - 5$.*

Theorem 2.5 *The group of automorphisms $\Gamma(G_n)$ is isomorphic to the dihedral group D_n of symmetries of a regular polygon of n sides.*

Theorem 2.6 *G_n is Hamiltonian for $n \geq 3$.*

Theorem 2.7 *The vertex-connectivity of G_n is equal to the minimum degree.*

3 General case

Our study when P is in general position has focused so far on properties of the center of $T(P)$ and of the minimum and maximum degree. In what follows $CH(P)$ denotes the convex hull of P, and S_p denotes the star at p.

Lemma 3.1 *Let P_n be a set of n points not in convex position and let T be in $T(P_n)$. If T is not a star, then $e(T) \geq n-1$.*

We denote by $CH_2(P_n)$ the convex hull of $P_n \setminus CH(P_n)$.

Theorem 3.2 *Let P_n be a set of n points. Then the radius of $T(P_n)$ is equal to $n-2$ and the center $\mathcal{C}$ of $T(P_n)$ satisfies*

$$\{S_p | p \in CH(P_n) \cup CH_2(P_n)\} \subset \mathcal{C} \subset \{S_p | p \in P_n\}.$$

In general, equalities do not hold.

Let P_n be a set of points with exactly two convex layers. A consequence of the above result is that the center of $T(P_n)$ is completely determined.

Corollary 3.3 *Let P_n be a set of points with exactly two convex layers. Then the center of $T(P_n)$ consists of all the stars.*

Recently, Rivera and Urrutia [5] showed that $2n-4$ is a lower bound for the minimum degree of $T(P)$ for any set P of n points, and that $\binom{n+1}{3} - n + 1$ is an upper bound for the maximum degree. Notice that both bounds are reached in the case where the points are in convex position.

Concerning *upper* bounds for the minimum degree and *lower* bounds for the maximum degree, we have obtained the following results.

Theorem 3.4 *Let P_n be a set of n points with k convex layers. Then the minimum degree $\delta(T(P_n))$ is at most $(4k-2)(n-1)$.*

This shows in particular that $\delta(T(P_n))$ is $O(n^2)$. On the other hand we have:

Theorem 3.5 *The maximum degree $\Delta(T(P_n))$ for a set P_n of n points is $\Omega(n^2)$.*

The question of finding the exact order of magnitude of $\max\{\delta(T(P)) : |P| = n\}$ and of $\min\{\delta(T(P)) : |P| = n\}$ remains open.

References

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