Euclidean Proximity and Power Diagrams

Leonidas J. Guibas* Li Zhang*

Computer Science Department
Stanford University
{guibas,lizhang}@cs.stanford.edu

1 Introduction

Power diagrams [Aur87] are a useful generalization of Voronoi diagrams in which the sites defining the diagram are not points but balls. They derive their name from the fact that the distance used in their definition is not the standard Euclidean distance, but instead the classical notion of the *power* of a point with respect to a ball [Cox61]. The advantage of the power distance is that the 'bisector surfaces', the surfaces equidistant from two given balls, are hyper-planes, and thus the Euclidean structure of these diagrams is polyhedral and they are easier to compute and manipulate than standard Voronoi diagrams. Several algorithms and applications for the power diagrams are already known, including those referenced in [IIM85, Aur88]. Most recently power diagrams have played a key role in work related to alpha shapes and 'skins' [Ede93, EM94, Ede95], as collections of balls are a natural model for molecular structures.

The Voronoi diagram of point sites in any dimension has many useful proximity properties. Let us say that two sites are neighbors in the Voronoi diagram if their Voronoi cells share a common facet. It is well known that, for a set of points S, the closest pair of points in S are neighbors in the Voronoi diagram of S — equivalently, the closest pair of points must define an edge of the dual Delaunay diagram. Even more is true: the nearest neighbor to any point $p \in S$ is a neighbor to p in the Voronoi diagram (equivalently, is connected to this neighbor by a Delaunay edge). Surprisingly, it seems that analogous proximity properties have not been studied for the power diagram. In this note, we consider the the power diagram of a set of disjoint balls. We show that the closest pair of balls (in Euclidean sense) are neighbors in the power diagram, while the nearest neighbor to a given ball is not necessarily a power diagram neighbor.

An application context in which these proximity issues arise is when we are trying to detect collisions among multiple moving objects. An approach to this problem is to surround each object by a bounding ball and then detect collisions between these balls. If we can continuously keep track of the closest pair of balls as the objects move, we will have a way to detect such collisions. This setting is examined in [BGZ97], where a kinetic data structure ([BGH97]) is given for the case when all the balls have similar sizes. However, that method doesn't apply to the case when there is no bound on the ratio on the largest radius to the smallest radius. The result of this note implies that another way to maintain the closest pair among a set of moving balls is to maintain (dual of) the power diagram of the balls. As long as the moving balls are disjoint, maintaining the power diagram is similar to maintaining the Voronoi diagram of moving points. In both cases local certificates prove the correctness of the structure; when one of them fails, an easy update restores the correctness of the structure.

2 Definitions and Notations

Before stating the main results, we first need some definitions and notations. For two points $p, q \in \Re^d$, denote by pq the line segment joining them. Let $|pq|$ denote the Euclidean distance between p, q. A d-ball $b(c, r)$, for $c \in \Re^d, r \geq 0$, is the point set $\{p \in \Re^d \mid |pc| \leq r\}$. The point c is the center of $b(c, r)$, and r is the radius. For a ball σ, denote by c_σ the center of σ and by r_σ its radius. A $(d-1)$-sphere is the boundary of a d-ball. The intersection between two $(d-1)$-spheres, if non-empty, is a $(d-2)$-sphere and lies on a hyper-plane that we call the *intersection plane* of the two spheres. For two d-balls σ_1, σ_2, we denote by $H_{\sigma_1\sigma_2}$ the intersection plane of their

*Leonidas Guibas and Li Zhang were supported in part by Army Research Office MURI grant 5-23542-A and NSF grant CCR-9623851.

boundaries. The intersection plane $H_{\sigma_1\sigma_2}$ is always perpendicular to the line passing through $c_{\sigma_1}, c_{\sigma_2}$. Further, in each of the half-spaces bounded by $H_{\sigma_1\sigma_2}$, either σ_1 is contained in σ_2, or σ_2 is contained in σ_1.

The *distance* of $p \in \Re^d$ from σ, $d_\sigma(p)$, is $|pc_\sigma| - r_\sigma$ if $p \notin \sigma$, and 0 otherwise. The *distance*, $d(\sigma_1, \sigma_2)$, between two disjoint d-balls σ_1, σ_2, is $|c_{\sigma_1}c_{\sigma_2}| - r_{\sigma_1} - r_{\sigma_2}$, which is the distance between two balls along the closest approach. Define $e_{\sigma_2}(\sigma_1) = b(c_{\sigma_1}, r_{\sigma_1} + d(\sigma_1, \sigma_2))$, i.e. the ball concentric to σ_1 and tangent to σ_2.

Consider a d-ball σ. Define the *power distance* of $p \in \Re^d \setminus \sigma$ from σ to be $\pi_\sigma(p) = \sqrt{|pc_\sigma|^2 - r_\sigma^2}$ — this is the square-root of the classical *power* of p with respect to σ. We use the current definition only for notational convenience. Throughout this note, we will never compute the power distance from a point to a ball containing the point. From the definitions, it is obvious that $\pi_\sigma(p) \geq d_\sigma(p)$. The *bisector* of two disjoint balls σ_1, σ_2 is the set of points which have equal power distance from σ_1 and σ_2. It is easy to verify that the bisector is a hyper-plane separating the two balls. Consider the intersection point, say o, between the bisector between σ_1, σ_2 and the line joining the centers $c_{\sigma_1}, c_{\sigma_2}$. We denote by $\delta_{\sigma_1\sigma_2}$ the ball $b(o, \pi_{\sigma_1}(o))$.

For a set of disjoint balls B, the *power diagram cell* of $\sigma \in B$ is $V_\sigma = \{p \in \Re^d \mid \pi_\sigma(p) \leq \pi_{\sigma'}(p), \sigma' \in B\}$. Each power diagram cell V_σ is a convex cell bounded by the bisectors between σ and every other ball in B. All the power diagram cells form a decomposition of $\Re^d$, which is the power diagram. Two power diagram cells are *adjacent* in the power diagram if they share a facet. Here, a facet is a $d-1$ flat. When two power diagram cells are adjacent, we also call the corresponding balls adjacent balls. Two balls σ_1, σ_2 are adjacent in the power diagram of B, if and only if there exists a point p on the bisector of σ_1, σ_2 so that $\pi_{\sigma_1}(p) < \pi_\sigma(p)$ for all $\sigma\,(\neq \sigma_1, \sigma_2) \in B$. Note that throughout this paper, we don't assume non-degeneracy. Thus, it is possible that two power diagram cells share a low dimensional face but not a facet. In this case, we don't regard them as adjacent in the power diagram.

3 Main Results

The Voronoi diagram of a set of points has many Euclidean proximity properties. It is well known that for a set of points P and $p \in P$, the nearest neighbor (in Euclidean sense) in P to p is p's neighbor in the Voronoi diagram of P, and thus the closest pair of P are neighbors in the Voronoi diagram. In fact, the Voronoi diagram of a set of point sites can be viewed as a special case of power digram in which all the balls have the same radius. In this paper, we investigate the above proximity properties for general power diagrams.

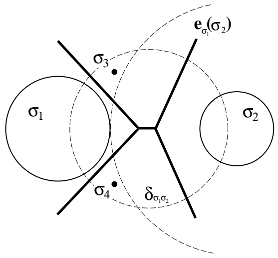

Figure 1: The power diagram of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. The segments are the boundaries of the power diagram cells. Two dashed circles are the boundaries of $\delta_{\sigma_1\sigma_2}$ and $e_{\sigma_1}(\sigma_2)$ respectively.

The main result of this paper is

Theorem 1 *For a set of disjoint d-balls, the closest pair, i.e. the pair of balls with the shortest Euclidean distance, are adjacent in the power diagram.*

Before proving the main theorem, we first construct an example to show that for a set of balls, the nearest neighbor to a ball is not necessarily a neighbor to it in the power diagram. Consider the situation as shown in Figure 1. There are four balls in the figure where σ_3, σ_4 are points (or tiny balls) inside the region $\delta_{\sigma_1\sigma_2} \setminus e_{\sigma_1}(\sigma_2)$ and lie on the different sides to the line passing through $c_{\sigma_1}, c_{\sigma_2}$. Recall that $e_{\sigma_1}(\sigma_2)$ is the ball concentric to c_{σ_1} and tangent to σ_2. Since σ_3, σ_4 are in the exterior of $e_{\sigma_1}(\sigma_2)$, their distances to σ_2 are greater than the distance between σ_1 and σ_2, i.e. σ_1 is the nearest neighbor to σ_2. However, σ_1, σ_2 are not neighbors in the power diagram because of the presence of σ_3 and σ_4.

Now, we proceed to prove the main theorem.

Proof: Suppose B is a set of disjoint balls, and $\sigma_1 = b(c_1, r_1), \sigma_2 = b(c_2, r_2)$ are the closest pair of balls in B. For simplicity of notation, we write $\sigma_1' = e_{\sigma_2}(\sigma_1)$, $\sigma_2' = e_{\sigma_1}(\sigma_2)$, and $\delta = \delta_{\sigma_1\sigma_2}$.

Let o be the intersection between the bisector of σ_1, σ_2 and the line segment c_1c_2. Let $r = \pi_{\sigma_1}(o)(\pi_{\sigma_2}(o))$. We will prove that there is no σ in B so that the power distance of o from σ is less than or equal to r, and thus conclude that σ_1, σ_2 are adjacent in the power diagram.

Recall that δ is the ball centered at o and with the radius r. If δ doesn't intersect any other balls

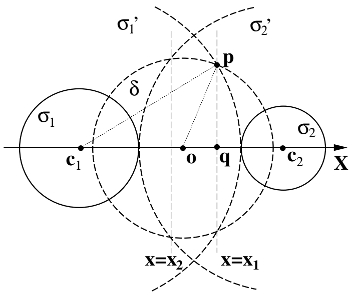

Figure 2: The proof

in B, then it follows that for any $\sigma \in B \setminus \{\sigma_1, \sigma_2\}$, $d_\sigma(o) > r$. Since for any σ, $\pi_\sigma(o) \geq d_\sigma(o)$, it follows that $\pi_\sigma(o) > r$. Therefore, it suffices to show that δ is free of any other balls in B.

Because σ_1, σ_2 are the closest pair of balls in B, for any $\sigma \in B \setminus \{\sigma_1, \sigma_2\}$, σ cannot intersect the interior of either σ'_1 or σ'_2. Thus, if δ is contained in the *interior* of $\sigma'_1 \cup \sigma'_2$, then we will be done. Without loss of generality, select a coordinate system by setting o to be the origin and the vector $\overrightarrow{c_1c_2}$ to be the direction of one axis, say the x-axis (Figure 2). Let $d_1 = |oc_1|$ and $d_2 = |oc_2|$. Consider the intersection planes $H_{\sigma'_1\delta}, H_{\sigma'_2\delta}$. Recall that the intersection plane is perpendicular to the line connecting the centers of the corresponding balls. Suppose that, in the coordinate system we chose, the equations defining $H_{\sigma'_1\delta}, H_{\sigma'_2\delta}$ are $x = x_1, x = x_2$, respectively. Observe that δ is contained in σ'_1 in the half-space to the left of $H_{\sigma'_1\delta}$, and contained in σ'_2 in the half-space to the right of $H_{\sigma'_2\delta}$. Therefore, it suffices to prove that $x_2 < x_1$ (Figure 2).

To compute x_1, consider a point $p \in \sigma'_1 \cap \delta$. Let q denote the intersection point between $H_{\sigma'_1\delta}$ and the x-axis (Figure 2). Then, we have $|pq|^2 = |c_1p|^2 - |c_1q|^2 = |op|^2 - |oq|^2$, i.e. $(d_1+d_2-r_2)^2 - (d_1+x_1)^2 = r^2 - x_1^2$.

Recall that $r = \pi_{\sigma_2}(o) = \sqrt{d_2^2 - r_2^2}$. Thus, $(d_1 + d_2 - r_2)^2 - (d_1 + x_1)^2 = d_2^2 - r_2^2 - x_1^2$, i.e. $x_1 = d_2 - r_2 - (d_2r_2 - r_2^2)/d_1$ after some simple manipulation.

Similarly, $x_2 = -(d_1 - r_1 - (d_1r_1 - r_1^2)/d_2)$.

Hence, $x_1 - x_2 = d_2 - r_2 - (d_2r_2 - r_2^2)/d_1 + d_1 - r_1 - (d_1r_1 - r_1^2)/d_2 = \frac{1}{d_1d_2}(d_1(d_2^2 + r_1^2) + d_2(d_1^2 + r_2^2) - (d_1 + d_2)(d_1r_1 + d_2r_2))(*)$.

Because $d_1^2 - r_1^2 = d_2^2 - r_2^2$, we have $d_1^2 + r_2^2 = d_2^2 + r_1^2 = (d_1^2+d_2^2+r_1^2+r_2^2)/2$. Plugging it in $(*)$, we obtain $x_1 - x_2 = \frac{1}{2d_1d_2}((d_1 + d_2)(d_1^2 + d_2^2 + r_1^2 + r_2^2 - 2d_1r_1 - 2d_2r_2)) = \frac{1}{2d_1d_2}((d_1+d_2)((d_1-r_1)^2 + (d_2-r_2)^2)) > 0$. The last inequality is strictly greater because all the balls are disjoint, and thus $d_1 > r_1$ and $d_2 > r_2$. Hence, $x_1 > x_2$, proving that δ is contained in the interior of $\sigma'_1 \cup \sigma'_2$. That is, $\delta \cap \sigma = \emptyset$, for any $\sigma \in B \setminus \{\sigma_1, \sigma_2\}$. According to the earlier reasoning, the power Voronoi cells of σ_1, σ_2 share a common facet, i.e. σ_1, σ_2 are neighbors in the power diagram.

□

Note that in the above proof, we don't assume nondegeneracy. Therefore, if there are more than one pair achieving the minimum distance, all those pairs have to be neighbors in the power diagram.

References

[Aur87] F. Aurenhammer. Power diagrams: properties, algorithms and applications. *SIAM J. Comput.*, 16:78–96, 1987.

[Aur88] F. Aurenhammer. Improved algorithms for discs and balls using power diagrams. *J. Algorithms*, 9:151–161, 1988.

[BGH97] J. Basch, L. Guibas, and J. Hershberger. Data structures for mobile data. In *Proc. 8th ACM-SIAM Sympos. Discrete Algorithms*, pages 747–756, 1997.

[BGZ97] J. Basch, L. Guibas, and L. Zhang. Proximity problems on moving points. In *Proc. 13th Annu. ACM Sympos. Comput. Geom.*, pages 344–351, 1997.

[Cox61] H. S. M. Coxeter. *Introduction to Geometry.* Wiley, New York, NY, 1961.

[Ede93] H. Edelsbrunner. The union of balls and its dual shape. In *Proc. 9th Annu. ACM Sympos. Comput. Geom.*, pages 218–231, 1993.

[Ede95] H. Edelsbrunner. Smooth surfaces for multi-scale shape representation. In *Proc. 15th Conf. Found. Softw. Tech. Theoret. Comput. Sci.*, volume 1026 of *Lecture Notes Comput. Sci.*, pages 391–412. Springer-Verlag, 1995.

[EM94] H. Edelsbrunner and E. P. Mücke. Three-dimensional alpha shapes. *ACM Trans. Graph.*, 13(1):43–72, January 1994.

[IIM85] H. Imai, M. Iri, and K. Murota. Voronoi diagrams in the Laguerre geometry and its applications. *SIAM J. Comput.*, 14:93–105, 1985.