Perspective Projections and Removal of Degeneracies

Francisco Gómez [*] *Ferran Hurtado* [†] *Toni Sellarès* [‡] *Godfried Toussaint* [§]

Abstract

In this paper we consider the following problems: (1) computing projections with distinct x-coordinates, (2) computing non-collinear projections, (3) computing non-cocircular projections, and (4) computing projections with non-parallel line segments. Here the word projection means perspective projection, a kind of projection that has received little attention in computational geometry. These problems are relevant to non-degeneracy assumptions and to the robustness of geometric algorithms.

1 Introduction

The use of orthogonal projections is quite common in computational geometry. However, perspective projections have received little attention and are interesting in their own right. In this paper, projection will mean perspective projection. Given an object in space, a point (the center of projection) and a plane, the perspective projection of that object from the center onto the plane is defined by the intersection of this plane with the half-rays emanating from the center and going through each point of the object. The topic of this paper is related to the removal of geometric degeneracies in the input data. Yap [6] has distinguished between *problem-induced* and *algorithm-induced* degeneracies. There are at least two methods to cope with the former: approximation [4] and perturbation [1], [2], [6]. For the latter, Gómez et at. [3] have studied several ways to remove them. Their method consists of performing a global rigid transformation on the input that will remove the algorithm-induced degeneracy, and once the solution to a problem has been obtained, perform the inverse transformation on the result. Here we look at the problem of removing such degeneracies by using perspective projections as the global transformation. For example, suppose we are given a set of points on the xy-plane which has four or more cocircular points. We are interested in finding a center of projection and a plane such that the projection of that set of points does not contain cocircular points. However, planes which are far away from the xy-plane are not of interest to us since they could change the set of points too much. We therefore look for planes close to the xy-plane. As a more general problem, we consider computing a non-cocircular perspective projection of an arbitrary set of points in space. For other degeneracy assumptions, we also solve the general version, i.e., finding a perspective projection of the object in space that exhibits a given property.

2 Projections with Distinct x-Coordinates

A perspective projection onto a given plane from a point in space is said to be a *projection with distinct x-coordinates* if there are no two projected points with the same x-coordinate. Without loss of generality, we can fix the plane of projection to be the xy-plane. If there are two points which determine a line parallel to the y-axis, then their projection on the xy-plane will have the same x-coordinate, whether the center of projection is at infinity (orthogonal projection) or at any other point (perspective projection). The first step of our algorithm is to detect if such a situation occurs, since the desired projection would not otherwise exist. Once this is done, a point outside of the forbidden regions of projection is computed (such forbidden regions are the planes going through two points whose intersections with the xy-plane yield lines parallel to the y-axis). Note that in order to find such a point it is not necessary to explicitly construct the forbidden regions. We thus have the following theorem.

Theorem 1 *A center of projection such that the projection of a set of points onto the xy-plane has no two*

[*] Departamento de Matemática Aplicada, U.P.M., Madrid, España (fmartin@eui.upm.es). Partially supported by Proyecto MEC-DGES-SEUID PB96-0005-C02-02

[†] Departamento de Matemática Aplicada, U.P.C., Barcelona, España (hurtado@ma2.upc.es). Partially supported by Projecte UPC PR9410 and Proyecto MEC-DGES-SEUID PB96-0005-C02-02

[‡] Institut d'Infomàtica i Aplicacions, Universitat de Girona, Girona, España (sellares@ima.udg.es). Partially supported by Proyecto MEC-DGES-SEUID PB96-0005-C02-02

[§] School of Computer Science, McGill University, Montreal, Quebec, Canada (godfried@mull.cs.mcgill.ca). Partially supported by NSERC-A9293 and FCAR-EQ1678

points with the same x-coordinate, if one exists, can be computed in $O(n \log n)$ time and $O(n)$ space.

3 Non-collinear Projections

In this section we are interested in computing a *non-collinear projection* for a set of points in space, i.e., a center of projection such that there are no three collinear points on the projection. If the original set contains three or more points which are collinear, any projection will also have them collinear. The first part of our algorithm checks that condition before continuing. If there is a line through three points of the set, the algorithm halts and determines that such a projection does not exist. Otherwise, a point not belonging to the forbidden regions is computed. For this problem the forbidden regions are the planes going through all triplets determined by three points of the set. By using a property of duality, it can be shown that it is not necessary to look at all the $O(n^3)$ forbidden planes to obtain the desired point.

Theorem 2 *Given a set S of n distinct points in space, computing a non-collinear projection, if one exists, can be done in $O(n^2)$ time and space.*

4 Non-cocircular Projections

In this section we show how to compute *non-cocircular projections* of a set of points in space. The forbidden regions for this problem consist of a set of surfaces which are determined by all subsets of four points. We will assume that the intersection of each such surface with the z-axis can be computed in $O(1)$ time. By using a dual map, looking at the $O(n^4)$ surfaces can be avoided. Picking a point outside of these forbidden regions can be done within the bounds given by the following theorem.

Theorem 3 *Given a set of n distinct points in space, a non-cocircular projection can be computed in $O(n^3)$ time and space in an appropriate model of computation.*

5 Non-parallel Projections

In this last section it is shown how to compute a projection of a set of distinct line segments in space onto the xy-plane such that no two line segments in the projection are parallel. If there are two line segments in the set which are parallel to each other and parallel to the xy-plane, then no center of projection admits a non-parallel projection. Our algorithm first detects if this situation occurs and if not, then it computes a center of projection. The idea is, again, to find a point which does not belong to the forbidden regions. Given two line segments in space, the set of forbidden centers of projection are formed by those planes which yield parallel projections of the line segments on the xy-plane. It can be shown that such a set of forbidden centers is a quadric in space. We will assume that the intersection of these quadrics with the z-axis can be computed in $O(1)$ time. Finding a point outside the forbidden quadrics can be done within the bounds according to the following theorem.

Theorem 4 *A center of non-parallel projection, if it exists, can be computed in $O(n \log n)$ time and linear space in an appropiate model of computation.*

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