Planar sets with few empty convex polygons

Adrian Dumitrescu

Department of Computer Science, Rutgers University,
Hill Center for the Mathematical Sciences, Busch Campus,
Piscataway, NJ 08855, (e-mail: dumitres@paul.rutgers.edu)

Abstract

A configuration of n points in general position in the plane is described which has less than $1.684n^2$ empty triangles, less than $2.132n^2$ empty convex quadrilaterals, less than $1.229n^2$ empty convex pentagons and less than $0.298n^2$ empty convex hexagons. This improves the constants from a previous construction given by Valtr.

1 Introduction

First we recall the following problem formulated by Erdős and Szekeres [ES35]: Let $f(n)$ be the smallest integer so that if there are $f(n)$ points in the plane no three on a line, then there are always n of them which form the vertices of a convex n-gon. It has been proved ([ES35], [ES60]) that $f(n)$ is well defined for all integer $n \geq 3$. A related problem was considered in [Erd75]: Let $g(n)$ be the smallest integer so that if there are $g(n)$ points in the plane , no three on a line, then there are always n of them which form the vertices of an empty convex n-gon. It has been proved that $g(n)$ exists for $n = 3, 4, 5$ ([Ha78]), does not exist for $n \geq 7$ ([Ho83]), and its existence for $n = 6$ is still open. Two other related problems are concerned with the estimation of the number of convex k-gons and the number of empty convex k-gons determined by a set of n points. This last problem will be the subject of our paper.

We call a set S, ($|S| \geq 3$) of points in the plane independent if no three are on a line. Denote by $f_k(S)$ the number of convex k-gons determined by a set S of points and let $f_k(n) = \min\{f_k(S) : S \subset \mathbb{R}^2 \text{ independent }, |S| = n\}$. Similarly, denote by $g_k(S)$ the number of empty convex k-gons determined by a set S of points and let $g_k(n) = \min\{g_k(S) : S \subset \mathbb{R}^2 \text{ independent }, |S| = n\}$.

Katchalski and Meir [KM87] proved that there is a constant $K < 200$ such that for any $n \geq 3$

$$\binom{n-1}{2} \leq g_3(n) \leq Kn^2$$

Horton [Ho83] constructed arbitrarily large sets of points H_n, for $n = 2^t$, $t \in \mathbb{N}$ having no empty convex 7-gons giving $g_k(n) = 0$ for $k \geq 7$. Bárány and Füredi [BF87] proved that

$$n^2 - O(n \log n) \leq g_3(n) \leq g_3(H_n) \leq 2n^2$$

$$\frac{1}{4}n^2 - O(n\ \log n) \leq g_4(n) \leq g_4(H_n) \leq 3n^2$$

$$\lfloor \frac{n}{10} \rfloor \leq g_5(n) \leq g_5(H_n) \leq 2n^2$$

$$g_6(n) \le g_6(H_n) \le \frac{1}{2}n^2$$
$$g_k(n) = g_k(H_n) = 0 \text{ for } k \ge 7$$

where for the upper bounds, $n = 2^t,\ t \in \mathbb{N}$; (the lower bound on $g_5(n)$ is implied by the result of Harboth [Ha78] and the last equality by the result of Horton [Ho83]). As remarked in [Va95] the construction can be modified for n s.t. $2^t < n < 2^{t+1},\ t \in \mathbb{N}$, showing that the above inequalities hold for any n. Using 4 transformed copies of Horton's set, Valtr ([Va95]) has improved the upper bounds:

$$g_3(n) \le 1.8n^2, \quad g_4(n) \le 2.42n^2$$
$$g_5(n) \le 1.46n^2, \quad g_6(n) \le \frac{1}{3}n^2$$

In Section 2, we describe a simple construction which gives upper bounds in between the previous two by Bárány and Fűredi, respectively Valtr. We only present it for the simplicity of counting; it uses 3 Horton sets:

For any $\epsilon > 0$

$$g_3(n) \le (\frac{15}{8} + \epsilon)n^2 \approx 1.875n^2, \quad g_4(n) \le (\frac{11}{4} + \epsilon)n^2 \approx 2.750n^2$$
$$g_5(n) \le (\frac{7}{4} + \epsilon)n^2 \approx 1.750n^2, \quad g_6(n) \le (\frac{7}{16} + \epsilon)n^2 \approx 0.437n^2$$

In Section 3, we show a construction giving the following better bounds:

$$g_3(n) \le \frac{3771}{2240}n^2 \approx 1.683n^2, \quad g_4(n) \le \frac{955}{448}n^2 \approx 2.132n^2$$
$$g_5(n) \le \frac{393}{320}n^2 \approx 1.228n^2, \quad g_6(n) \le \frac{666}{2240}n^2 \approx 0.297n^2$$

In Section 4, we review two different proofs of a lower bound on $g_4(n)$, a result already known before ([DEO90], [Va95]):

$$g_4(n) \ge \frac{1}{2}n^2 - O(n\ \log n)$$

2 A simple first construction

We first recall Horton's construction, as presented in [BF87]: A point set $H(n)$ is defined inductively when n is a power of 2. In $H(n)$ all points have positive coordinates and the set of x-coordinates is $\{1, \dots, n\}$. To start with let $H(1) = \{(1,1)\}$ and $H(2) = \{(1,1),(2,2)\}$. When $H(n)$ is defined, set

$$H(2n) = \{(2x-1, y) : (x,y) \in H(n)\} \cup \{(2x, y + d_n) : (x,y) \in H(n)\}$$

where d_n is a large number, e.g., $d_n = 3^n$ is sufficient. As shown in [Va95], one can define such sets for an arbitrary n: let n' be the least power of 2, such that $n' \ge n$. Set

$$H(n) = \{(x,y) \in H(n') : x \le n\}$$

All y-coordinates of points in $H(n)$ are smaller than 3^n. For a given $\epsilon_0 > 0$, let $\gamma = \frac{1}{2} - \epsilon_0,\ \delta \le \delta_0 = 10^{-2}\epsilon_0 n^{-1} 3^{-n}$. Let $Q(n,\delta)$ be a "y-compressed" version of $H(n)$:

$$Q(n,\delta) = \{(x, \delta y) : (x,y) \in H(n)\}$$

The points of $Q(n, \delta)$ lie close to the first n positive lattice points of the x-axis $\{(1,0), \ldots, (n,0)\}$. Similarly, we define $R(n, \delta)$, another "compressed" version of $H(n)$ having "double density":

$$R(n, \delta) = \{(\gamma x, \delta y) : (x, y) \in H(n)\}, \text{ where } \delta \leq \delta_0$$

For any set P, $P + (u, v)$ denotes the set P shifted by the vector (u, v). Finally we construct an n-point set $S(n)$ in the following way:

$$S(n) = S_a \cup S_b \cup S_c$$

where

$$S_a = Q(\gamma_1 \frac{n}{4}, \delta_1) + (0, 2)$$

$$S_b = R(\gamma_2 \frac{n}{2}, \delta_2) + (\frac{1}{9}, 1)$$

$$S_c = Q(\gamma_1 \frac{n}{4}, \delta_3)$$

$$\gamma_1 = \frac{1 - 2\epsilon_0}{1 - \epsilon_0} \to 1 \text{ as } \epsilon_0 \to 0, \quad \gamma_2 = \frac{1}{1 - \epsilon_0} \to 1 \text{ as } \epsilon_0 \to 0 \tag{1}$$

Observation 1 *The set $S(n)$ has no three collinear points for some choice of δ_i, $\delta_i \leq \delta_0$, $i = 1, 2, 3$.*

(this can be seen since keeping δ_1, δ_2 fixed, δ_3 can be continuously modified in the allowed interval to achieve this property.)

Observation 2 *There are no empty triangles of the form aac or cca.*

(this follows from the choice of γ and δ_i, $(i = 1, 2, 3)$, δ_i sufficiently small compared to ϵ_0; since ϵ_0=fixed, $\gamma < \frac{1}{2}$ is fixed; a segment determined by any two consecutive points of S_a or any two consecutive points of S_c can be made an arbitrary close approximation of a horizontal segment of unit length by the choice of δ_1 and δ_3. Similarly a segment determined by any two consecutive points of S_b can be made an arbitrary close approximation of a horizontal segment of length $= \gamma$ by the choice of δ_2.)

Let S be a point set in the plane and two points u, v in S. The segment uv is said to be empty from below if when translated in the negative direction of the y-axis does not hit other points in S. A segment empty from above is defined similarly. A subset X of S is called open from below if all the line segments connecting two points of X are empty from below. Analogously, we define subsets of S empty from above. For an integer $r \geq 2$,denote by $H_r^-(S)$ the collection of r-subsets of S empty from below and by $h_r^-(S)$ their number; similarly by $H_r^+(S)$ the collection of r-subsets of S empty from above and by $h_r^+(S)$ their number. Bárány and Füredi [BF87] showed

$$h_2^-(H(n)) < 2n, \quad h_2^+(H(n)) < 2n$$

$$h_3^-(H(n)) < n, \quad h_3^+(H(n)) < n$$

$$h_r^-(Q^+(n)) = h_r^+(Q^-(n)) = 0 \text{ for } r \geq 4$$

Let $N_{\alpha\beta\gamma}$ denote the number of empty triangles having one point in S_α, one in S_β and one in S_γ. For simplicity, we assume that $|S_a| = |S_c| = \frac{n}{4}$ and $|S_b| = \frac{n}{2}$ and adjust only the final results according to (1): $\gamma_1, \gamma_2 \to 1$ as $\epsilon_0 \to 0$.

As in [BF87], we have:

$$N_{aaa} \le 2 \cdot \frac{n}{4} \cdot \frac{n}{4} = \frac{2}{16}n^2, \quad N_{bbb} \le 2 \cdot \frac{n}{2} \cdot \frac{n}{2} = \frac{8}{16}n^2, \quad N_{ccc} \le \frac{2}{16}n^2$$

$$N_{aab} \le h_2^-(S_a) \cdot |S_b| \le 2 \cdot \frac{n}{4} \cdot \frac{n}{2} = \frac{4}{16}n^2, \quad N_{bba} \le h_2^+(S_b) \cdot |S_a| \le 2 \cdot \frac{n}{2} \cdot \frac{n}{4} = \frac{4}{16}n^2$$

$$N_{bbc} \le \frac{4}{16}n^2, \quad N_{ccb} \le \frac{4}{16}n^2, \quad N_{abc} \le 2 \cdot \frac{n}{4} \cdot \frac{n}{4} = \frac{2}{16}n^2$$

The total number of empty triangles with the appropriate correction implied by (1) is:

$$N \le (\frac{2+8+2+4+4+4+4+2}{16} + \epsilon)n^2 = (\frac{15}{8} + \epsilon)n^2 \approx 1.875n^2$$

with $\epsilon = \epsilon(\epsilon_0) > 0$, arbitrarily small.

We continue by counting empty convex 4-gons. Similarly let $N_{\alpha\beta\gamma\delta}$ denote the number of empty 4-gons where the indices indicate from which sets the points are chosen. The only nonzero terms are bounded as follows:

$$N_{aaaa} \le 3 \cdot \frac{n}{4} \cdot \frac{n}{4} = \frac{3}{16}n^2, \quad N_{bbbb} \le 3 \cdot \frac{n}{2} \cdot \frac{n}{2} = \frac{12}{16}n^2, \quad N_{cccc} \le \frac{3}{16}n^2$$

$$N_{aaab} \le h_3^-(S_a) \cdot |S_b| \le \frac{n}{4} \cdot \frac{n}{2} = \frac{2}{16}n^2, \quad N_{abbb} \le |S_a| \cdot h_3^+(S_b) \le \frac{n}{4} \cdot \frac{n}{2} = \frac{2}{16}n^2$$

$$N_{aabb} \le h_2^-(S_a) \cdot h_2^+(S_b) \le 2 \cdot \frac{n}{4} \cdot 2 \cdot \frac{n}{2} = \frac{8}{16}n^2$$

$$N_{bbbc} \le \frac{2}{16}n^2, \quad N_{bbcc} \le \frac{8}{16}n^2, \quad N_{bccc} \le \frac{2}{16}n^2, \quad N_{abbc} \le \frac{n}{4} \cdot \frac{n}{4} = \frac{1}{16}n^2$$

The total number of empty 4-gons with the appropriate correction is:

$$N \le (\frac{3+12+3+2+8+2+2+8+2+1}{16} + \epsilon)n^2 = (\frac{43}{16} + \epsilon)n^2 \approx 2.687n^2$$

Next, counting the empty convex 5-gons, the only nonzero terms are:

$$N_{aaaaa} + N_{bbbbb} + N_{ccccc} \le 2 \cdot (\frac{1}{16}n^2 + \frac{4}{16}n^2 + \frac{1}{16}n^2) = \frac{12}{16}n^2$$

$$N_{aaabb} + N_{aabbb} \le 2 \cdot \frac{n}{4} \cdot \frac{n}{2} + 2 \cdot \frac{n}{2} \cdot \frac{n}{4} = \frac{8}{16}n^2, \quad N_{bbbcc} + N_{bbccc} \le \frac{8}{16}n^2$$

The total number of empty 5-gons with the appropriate correction is:

$$N \le (\frac{12+8+8}{16} + \epsilon)n^2 = (\frac{7}{4} + \epsilon)n^2 \approx 1.750n^2$$

Finally, in the case of 6-gons, the only nonzero terms are:

$$N_{aaaaaa} + N_{bbbbbb} + N_{cccccc} \le \frac{1}{2} \cdot (\frac{1}{16}n^2 + \frac{4}{16}n^2 + \frac{1}{16}n^2) = \frac{3}{16}n^2$$

$$N_{aaabbb} \le \frac{n}{4} \cdot \frac{n}{2} = \frac{2}{16}n^2, \quad N_{bbbccc} \le \frac{n}{2} \cdot \frac{n}{4} = \frac{2}{16}n^2$$

The total number of empty 6-gons with the appropriate correction is:

$$N \le (\frac{3+2+2}{16} + \epsilon)n^2 = (\frac{7}{16} + \epsilon)n^2 \approx 0.437n^2$$

3 A second construction

We use the notations introduced in Section 2 for the sets $H(n), Q(n,\delta)$ but the constants γ, δ are redefined. Using the recursive definition of the Horton set $H(n)$, for $n = 8$, we get:

$$H(8) = \{(1,1),(2,82),(3,10),(4,91),(5,2),(6,83),(7,11),(8,92)\}$$

Let h_i be the y-coordinate of the i-th point in left to right order from $H(8)$, where counting starts from 0. Set $\delta = 10^{-6}n^{-1}3^{-n}, \gamma = 10^{-4}$. We construct an n-point set $S(n)$ in the following way:

$$S(n) = \cup_{i=0}^{i=7} S_i$$

where

$$Q(n,\delta) = \{(x, \delta y) : (x,y) \in H(n)\}$$
$$S_i = Q(\frac{n}{8}, \delta) + (\gamma h_i, i) \quad i = 0, \dots, 7$$

The set described above is an approximation of an $8 \times \frac{n}{8}$ rectangular subset of the square grid.

Observation 3 *The set $S(n)$ has no three collinear points.*

(this follows from the properties of the Horton set and from the choice of δ and γ, δ sufficiently small compared to γ.)

Let P be an empty convex polygon in $S(n)$ with

$$i_1(P) = min\{i : P \cap S_i \neq \emptyset\}, \quad i_2(P) = max\{i : P \cap S_i \neq \emptyset\}, \quad \Delta(P) = i_2(P) - i_1(P)$$

We fix $k \in \{3,4,5,6\}$, the size of the empty polygons we intend to count, say N is their total number. For $\Delta = 0,1,2,3,4,5,6,7$, let N_Δ be the number of empty k-gons P in $S(n)$, with $\Delta(P) = \Delta$. Then

$$N = \sum_{i=0}^{7} N_\Delta$$

Consider a segment s determined by a pair of points in $S(n)$ having the two endpoints in S_i and S_j with $i < j$ and let $\Delta = j - i$. For $i = 0, \dots, 7$, denote the j-th point of S_i in left to right order by $p_{ij}, \quad j = 1, \dots, \frac{n}{8}$

Observation 4 *Let $s = p_{i,l}p_{j,m}$ be a segment in $S(n)$ and $\Delta = j - i \geq 2$. For $m + \Delta \leq \frac{n}{8}$ consider the segment $s' = p_{i,l}p_{j,m+\Delta}$. Then s and s' determine the same number of empty triangles having a third point in S_k, for some k, $i < k < j$.*

(for i, j, l, m fixed, there is a 1-1 correspondence between the set of empty triangles of the form $t = p_{i,l}p_{j,m}p_{k,r}$ and the set of empty triangles of the form $t' = p_{i,l}p_{j,m+\Delta}p_{h,s}$, $i < k, h < j$, $1 \leq r, s \leq \frac{n}{8}$.)

Observation 5 *There is no empty triangle of the form $p_{i,l}p_{i,m}p_{j,r}$ with $\Delta = |j - i| \geq 2$ and $|m - l| \geq 2$.*

(this is clearly true for $\Delta \geq 3$ and for $\Delta = 2$ is implied by the same 1-1 correspondence.)

Observation 6 *Let* $t = p_{i,l}p_{i,l+1}p_{j,m}$ *be a triangle in* $S(n)$ *and* $\Delta = |j-i| \geq 2$. *For* $m + \Delta \leq \frac{n}{8}$ *consider the triangle* $t' = p_{i,l}p_{i,l+1}p_{j,m+\Delta}$. *Then* t *is empty if and only if* t' *is empty.*

(there is a 1-1 correspondence between the interior points of the triangles t and t'.)

The above observations imply that we can restrict the counting of empty polygons to a few cases.

For $\Delta = 2, 3, 4, 5, 6, 7$, we denote by (the same notation will be used for different values of k):

A'_Δ=average number of empty k-gons P determined by a segment $s = p_{i,l}p_{j,m}$, $i = i_1(P)$, $j = i_2(P)$, $\Delta = j - i \geq 2$, where P intersects each of S_{i_1} and S_{i_2} in exactly one point ($i_1(P)$ and $i_2(P)$ were defined previously; for a fixed pair $\{i, j\}$ there are $\frac{1}{64}n^2$ such segments).

A''_Δ=average number of empty k-gons P determined by a triangle $t = p_{i,l}p_{i,l+1}p_{j,m}$, $\Delta = |j-i| \geq 2$, $\{i, j\} = \{i_1(P), i_2(P)\}$, where P intersects S_{i_1} or S_{i_2} in at least two points (for a fixed pair $\{i, j\}$ there are at most $\frac{2}{64}n^2$ such triangles).

N'_Δ=total number of empty k-gons P, where P intersects each of S_{i_1} and S_{i_2} in exactly one point.

N''_Δ=total number of empty k-gons P, where P intersects S_{i_1} or S_{i_2} in at least two points.
These imply:

$$N'_\Delta \leq A'_\Delta \cdot (8 - \Delta) \cdot \frac{n^2}{64}$$

$$N''_\Delta \leq A''_\Delta \cdot (8 - \Delta) \cdot \frac{2 \cdot n^2}{64}$$

$$N_\Delta = N'_\Delta + N''_\Delta$$

Figures 1 through 9 may be used to check the following calculations.
We start with $k = 3$:

$$A'_2 \leq 2, \quad A''_2 \leq \frac{4}{8} = \frac{1}{2}, \quad A'_3 \leq \frac{4+2+2}{3} = \frac{8}{3}, \quad A''_3 = 0$$

$$A'_4 \leq \frac{3+3+3+4}{5} = \frac{13}{4}, \quad A''_4 \leq \frac{2}{8} \cdot \frac{1}{2} = \frac{1}{8}$$

$$A'_5 \leq \frac{(3+5+3+2+3)+(2+2+4+4+4)+(3+5+3+2+3)}{3 \cdot 5} = \frac{16}{5}, \quad A''_5 = 0$$

$$A'_6 \leq \frac{3+4+2+3+3+2}{6} = \frac{17}{6}, \quad A''_6 = 0$$

$$A'_7 \leq \frac{2+2+6+2+2+4+4}{7} = \frac{22}{7}, \quad A''_7 = 0$$

$$N_0 \leq 8 \cdot 2 \cdot \frac{n}{8} \cdot \frac{n}{8} = 16 \cdot \frac{n^2}{64}, \quad N_1 \leq 7 \cdot 2 \cdot 2 \cdot \frac{n}{8} \cdot \frac{n}{8} = 28 \cdot \frac{n^2}{64}$$

$$N_2 \leq 6 \cdot (2+1) \cdot \frac{n^2}{64} = 18 \cdot \frac{n^2}{64}, \quad N_3 \leq 5 \cdot \frac{8}{3} \cdot \frac{n^2}{64} = \frac{40}{3} \cdot \frac{n^2}{64}$$

$$N_4 \le 4 \cdot (\frac{13}{4} + \frac{1}{4}) \cdot \frac{n^2}{64} = 14 \cdot \frac{n^2}{64}, \quad N_5 \le 3 \cdot \frac{16}{5} \cdot \frac{n^2}{64} = \frac{48}{5} \cdot \frac{n^2}{64}$$

$$N_6 \le 2 \cdot \frac{17}{6} \cdot \frac{n^2}{64} = \frac{17}{3} \cdot \frac{n^2}{64}, \quad N_7 \le 1 \cdot \frac{22}{7} \cdot \frac{n^2}{64} = \frac{22}{7} \cdot \frac{n^2}{64}$$

$$N = \sum_{i=0}^{7} N_\Delta \le (16 + 28 + 18 + \frac{40}{3} + 14 + \frac{48}{5} + \frac{17}{3} + \frac{22}{7}) \cdot \frac{n^2}{64} = \frac{3771}{2240} n^2 \approx 1.683 n^2$$

We continue with $k = 4$:

$$A_2' \le 1, \quad A_2'' \le \frac{8}{8} = 1, \quad A_3' \le \frac{\binom{4}{2} + \binom{2}{2} + \binom{2}{2}}{3} = \frac{8}{3}, \quad A_3'' = 0$$

$$A_4' \le \frac{\binom{3}{2} + \binom{3}{2} + \binom{3}{2} + \binom{4}{2}}{4} = \frac{15}{4}, \quad A_4'' \le \frac{4}{8} \cdot \frac{1}{2} = \frac{1}{4}$$

$$A_5' \le \frac{(\binom{3}{2} + \binom{5}{2} + \binom{3}{2} + \binom{2}{2} + \binom{3}{2}) + (\binom{2}{2} + \binom{2}{2} + \binom{4}{2} + \binom{4}{2} + \binom{4}{2})}{2 \cdot 5} = \frac{40}{10} = 4, \quad A_5'' = 0$$

$$A_6' \le \frac{\binom{3}{2} + \binom{4}{2} + \binom{2}{2} + \binom{3}{2} + \binom{3}{2} + \binom{2}{2}}{6} = \frac{17}{6}, \quad A_6'' = 0$$

$$A_7' \le \frac{\binom{2}{2} + \binom{2}{2} + \binom{6}{2} + \binom{2}{2} + \binom{2}{2} + \binom{4}{2} + \binom{4}{2}}{7} = \frac{31}{7}, \quad A_7'' = 0$$

$$N_0 \le 8 \cdot 3 \cdot \frac{n}{8} \cdot \frac{n}{8} = 24 \cdot \frac{n^2}{64}, \quad N_1 \le 7 \cdot 3 \cdot 2 \cdot \frac{n}{8} \cdot \frac{n}{8} = 42 \cdot \frac{n^2}{64}$$

$$N_2 \le 6 \cdot (1 + 2) \cdot \frac{n^2}{64} = 18 \cdot \frac{n^2}{64}, \quad N_3 \le 5 \cdot \frac{8}{3} \cdot \frac{n^2}{64} = \frac{40}{3} \cdot \frac{n^2}{64}$$

$$N_4 \le 4 \cdot (\frac{15}{4} + \frac{2}{4}) \cdot \frac{n^2}{64} = 17 \cdot \frac{n^2}{64}, \quad N_5 \le 3 \cdot 4 \cdot \frac{n^2}{64} = 12 \cdot \frac{n^2}{64}$$

$$N_6 \le 2 \cdot \frac{17}{6} \cdot \frac{n^2}{64} = \frac{17}{3} \cdot \frac{n^2}{64}, \quad N_7 \le 1 \cdot \frac{31}{7} \cdot \frac{n^2}{64} = \frac{31}{7} \cdot \frac{n^2}{64}$$

$$N = \sum_{i=0}^{7} N_\Delta \le (24 + 42 + 18 + \frac{40}{3} + 17 + 12 + \frac{17}{3} + \frac{31}{7}) \cdot \frac{n^2}{64} = \frac{955}{448} n^2 \approx 2.132 n^2$$

We continue with $k = 5$:

$$A_2' = 0, \quad A_2'' \le \frac{4}{8} = \frac{1}{2}, \quad A_3' \le \frac{\binom{4}{3}}{3} = \frac{4}{3}, \quad A_3'' = 0$$

$$A_4' \le \frac{\binom{3}{3} + \binom{3}{3} + \binom{3}{3} + \binom{4}{3}}{4} = \frac{7}{4}, \quad A_4'' \le \frac{2}{8} \cdot \frac{1}{2} = \frac{1}{8}$$

$$A'_5 \leq \frac{2\cdot(\binom{3}{3}+\binom{5}{3}+\binom{3}{3}+\binom{3}{3})+1\cdot(\binom{4}{3}+\binom{4}{3}+\binom{4}{3})}{3\cdot 5} = \frac{2\cdot 13+12}{3\cdot 5} = \frac{38}{15}, \quad A''_5 = 0$$

$$A'_6 \leq \frac{\binom{3}{3}+\binom{4}{3}+\binom{3}{3}+\binom{3}{3}}{6} = \frac{7}{6}, \quad A''_6 = 0, \quad A'_7 \leq \frac{\binom{6}{3}+\binom{4}{3}+\binom{4}{3}}{7} = \frac{28}{7} = 4, \quad A''_7 = 0$$

$$N_0 \leq 8\cdot 2\cdot \frac{n}{8}\cdot\frac{n}{8} = 16\cdot\frac{n^2}{64}, \quad N_1 \leq 7\cdot 2\cdot 2\cdot\frac{n}{8}\cdot\frac{n}{8} = 28\cdot\frac{n^2}{64}$$

$$N_2 \leq 6\cdot 1\cdot\frac{n^2}{64} = 6\cdot\frac{n^2}{64}, \quad N_3 \leq 5\cdot\frac{4}{3}\cdot\frac{n^2}{64} = \frac{20}{3}\cdot\frac{n^2}{64}$$

$$N_4 \leq 4\cdot(\frac{7}{4}+\frac{1}{4})\cdot\frac{n^2}{64} = 8\cdot\frac{n^2}{64}, \quad N_5 \leq 3\cdot\frac{38}{15}\cdot\frac{n^2}{64} = \frac{38}{5}\cdot\frac{n^2}{64}$$

$$N_6 \leq 2\cdot\frac{7}{6}\cdot\frac{n^2}{64} = \frac{7}{3}\cdot\frac{n^2}{64}, \quad N_7 \leq 1\cdot 4\cdot\frac{n^2}{64} = 4\cdot\frac{n^2}{64}$$

$$N = \sum_{i=0}^{7} N_\Delta \leq (16+28+6+\frac{20}{3}+8+\frac{38}{5}+\frac{7}{3}+4)\cdot\frac{n^2}{64} = \frac{393}{320}n^2 \approx 1.228n^2$$

We end with $k = 6$:

$$A'_2 = 0, \quad A''_2 = 0, \quad A'_3 \leq \frac{\binom{4}{4}}{3} = \frac{1}{3}, \quad A''_3 = 0, \quad A'_4 \leq \frac{\binom{4}{4}}{4} = \frac{1}{4}, \quad A''_4 = 0$$

$$A'_5 \leq \frac{2\cdot\binom{5}{4}+1\cdot(\binom{4}{4}+\binom{4}{4}+\binom{4}{4})}{3\cdot 5} = \frac{2\cdot 5+3}{3\cdot 5} = \frac{13}{15}, \quad A''_5 = 0$$

$$A'_6 \leq \frac{\binom{4}{4}}{6} = \frac{1}{6}, \quad A''_6 = 0, \quad A'_7 \leq \frac{\binom{6}{4}+\binom{4}{4}+\binom{4}{4}}{7} = \frac{17}{7}, \quad A''_7 = 0$$

$$N_0 \leq 8\cdot\frac{1}{2}\cdot\frac{n}{8}\cdot\frac{n}{8} = 4\cdot\frac{n^2}{64}, \quad N_1 \leq 7\cdot 1\cdot\frac{n}{8}\cdot\frac{n}{8} = 7\cdot\frac{n^2}{64}$$

$$N_2 = 0, \quad N_3 \leq 5\cdot\frac{1}{3}\cdot\frac{n^2}{64} = \frac{5}{3}\cdot\frac{n^2}{64}$$

$$N_4 \leq 4\cdot\frac{1}{4}\cdot\frac{n^2}{64} = 1\cdot\frac{n^2}{64}, \quad N_5 \leq 3\cdot\frac{13}{15}\cdot\frac{n^2}{64} = \frac{13}{5}\cdot\frac{n^2}{64}$$

$$N_6 \leq 2\cdot\frac{1}{6}\cdot\frac{n^2}{64} = \frac{1}{3}\cdot\frac{n^2}{64}, \quad N_7 \leq 1\cdot\frac{17}{7}\cdot\frac{n^2}{64} = \frac{17}{7}\cdot\frac{n^2}{64}$$

$$N = \sum_{i=0}^{7} N_\Delta \leq (4+7+0+\frac{5}{3}+1+\frac{13}{5}+\frac{1}{3}+\frac{17}{7})\cdot\frac{n^2}{64} = \frac{666}{2240}n^2 \approx 0.297n^2$$

Remark. At the end of [Va95], it is claimed that a point set A_n, of n points in general position described in [Va92] contains less than $1.68n^2$ empty triangles for large n. However it is said that the derivation of this fact is involved and no proof is given. We think that obtaining estimates for a the analogous construction approximating the $t \times \frac{n}{t}$ grid either with $t = 16$ or with $t = \sqrt{n}$ will give further improvement of these upper bounds.

4 A note on the lower bound for empty 4-gons

In [DEO90] it has been shown that the following inequality holds for an independent set S:

$$g_4(S) \geq g_3(S) - \binom{n-1}{2}$$

Combining this with the lower bound on $g_3(n)$ gives a lower bound on $g_4(n)$:

$$g_4(n) \geq \frac{1}{2}n^2 - O(n \ \log n) \tag{2}$$

As noted in [Va95], a similar lower bound was found by Bárány:

$$g_4(n) \geq \frac{1}{2}n^2 - O(n)$$

We present here a proof of (2) based on ideas from [BF87], including the next lemma on graphs:

Lemma. *Let G be a graph on the vertex set $\{1, 2, \ldots, n\}$. Suppose that there exist no four vertices $i < j < k < l$ such that (i, l), (i, k) and $(j, l) \in E(G)$. Then $|E(G)| \leq 3n\lceil \log_2 n \rceil$.*

Consider a point set S of size n, no three collinear. Let l be a line such that all projections of the points on l are distinct, say l is horizontal. Let $q_1, \ldots, q_n$ be the projection points of the set on the line in left to right ordering, where q_i is the projection of p_i on l. Let G_u and G_d be two graphs on the vertex set $\{q_1, \ldots, q_n\}$ such that

$E(G_u) = \{(q_i q_j)$: $i < j$ and for $i < k < j$, every p_k is below the segment $[p_i p_j]$ and at most one of the triangles $p_i p_k p_j$ is empty $\}$

$E(G_d) = \{(q_i q_j)$: $i < j$ and for $i < k < j$, every p_k is above the segment $[p_i p_j]$ and at most one of the triangles $p_i p_k p_j$ is empty $\}$

It has been shown in [BF87] that G_u and G_d satisfy the conditions of the previous lemma, hence $|E(G_u)|, |E(G_d)| = O(n \ \log n)$. Let $E = \{(q_i, q_j) : \ i < j\}$ and $E' = E - E(G_u) - E(G_d)$. Then $|E'| = \binom{n}{2} - O(n \ \log n)$. We say that an empty convex 4-gon is "determined" by the segment $[p_i p_j]$, where $i < j$ if its other two vertices have indices k, l such that $i < \{k, l\} < j$. Every segment $[p_i p_j]$, with $i < j, (q_i q_j) \in E'$ determines (at least) two empty triangles $p_i p_j p_k$ and $p_i p_j p_l$ where $i < \{k, l\} < j$. If the two triangles lie on opposite sides of the segment $[p_i p_j]$, we already have an empty convex 4-gon determined by $[p_i p_j]$; if they lie on the same side and say, $k < l$, the 4-gon $p_i p_k p_l p_j$ is convex (figure 10). Let o be the intersection of $[p_i p_l]$ with $[p_k p_j]$. If $p_i p_k p_l p_j$ is not empty, so the triangle $o p_k p_l$ is not empty, by choosing a point $p_{k'}$ inside this triangle, we get a new convex 4-gon $p_i p_k p_l p_j$ with $k = k', i < k < l < j$, for which the triangle $o p_k p_l$ has fewer points inside. We can continue this procedure until we obtain an empty convex 4-gon determined by the segment $[p_i p_j]$. Since every segment $[p_i p_j]$, with $i < j, (q_i q_j) \in E'$ determines an empty convex 4-gon, their number is $\geq |E'| = \frac{1}{2}n^2 - O(n \log n)$, as desired.

Acknowledgments. I would like to thank William Steiger for bringing this problem to my attention and for his suggestions and reviewing during my work on this paper. I also thank Pavel Valtr for helpful remarks.

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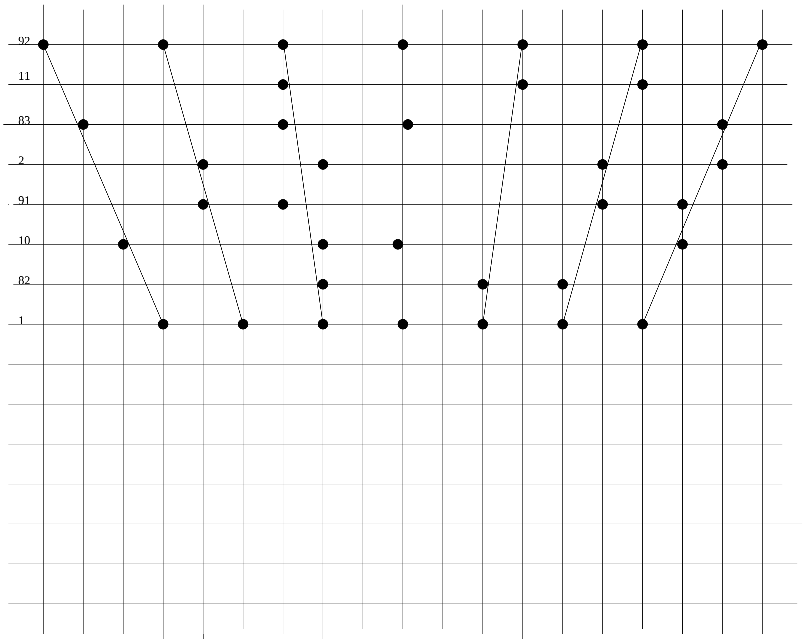

Figure 1: Delta=7

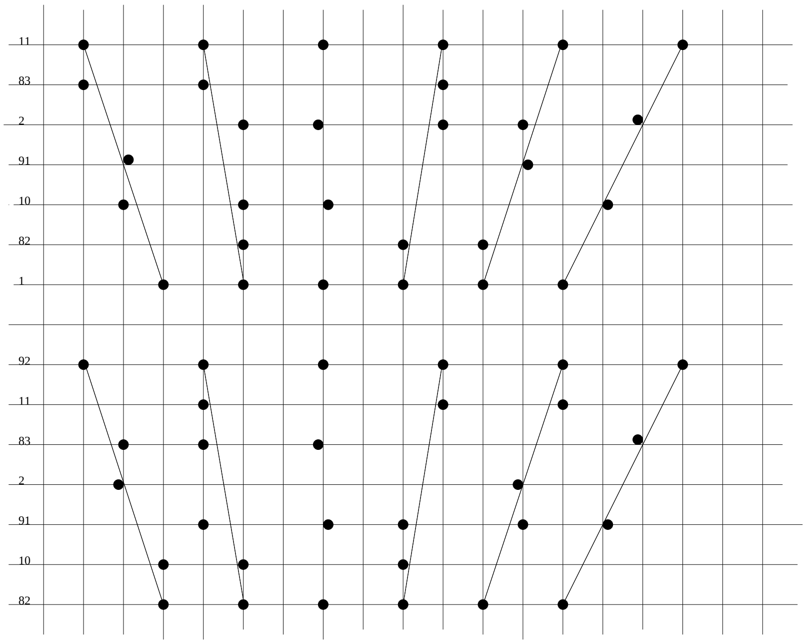

Figure 2: Delta=6

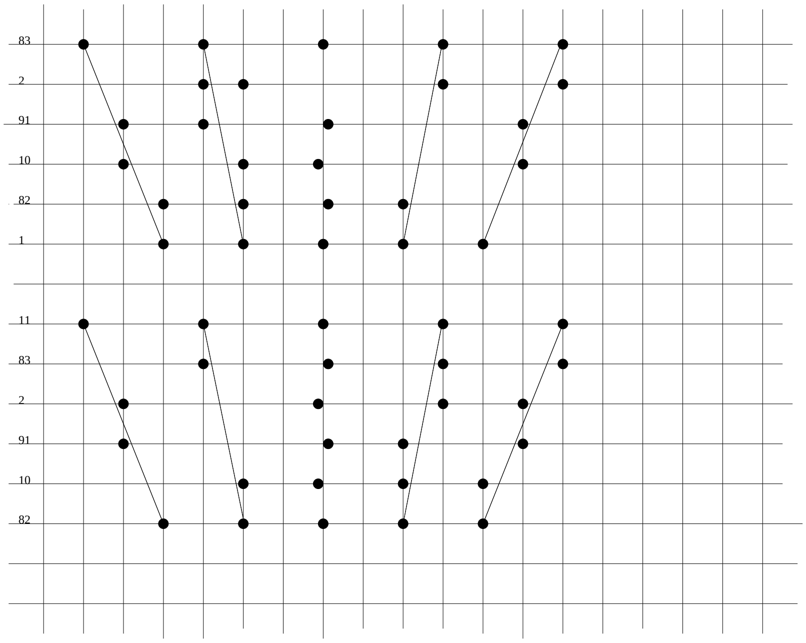

Figure 3: Delta=5

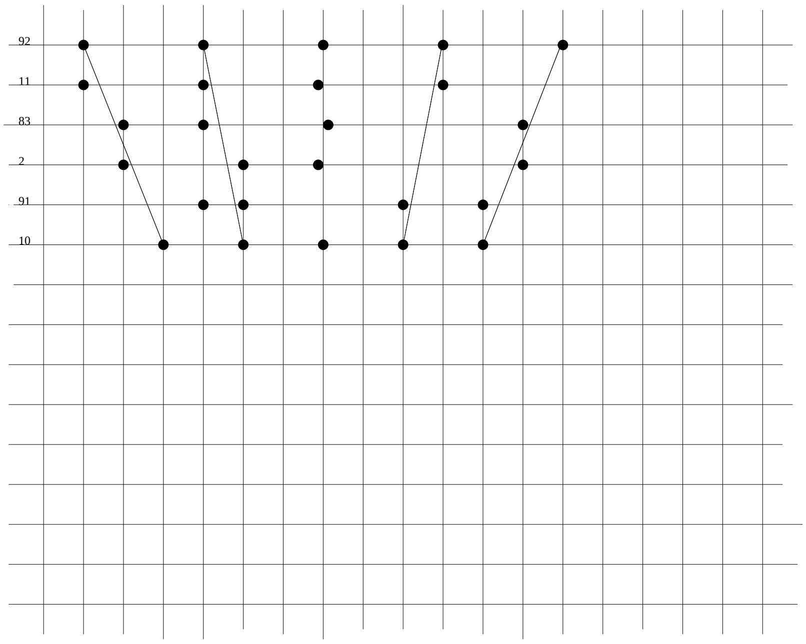

Figure 4: Delta=5 (continued)

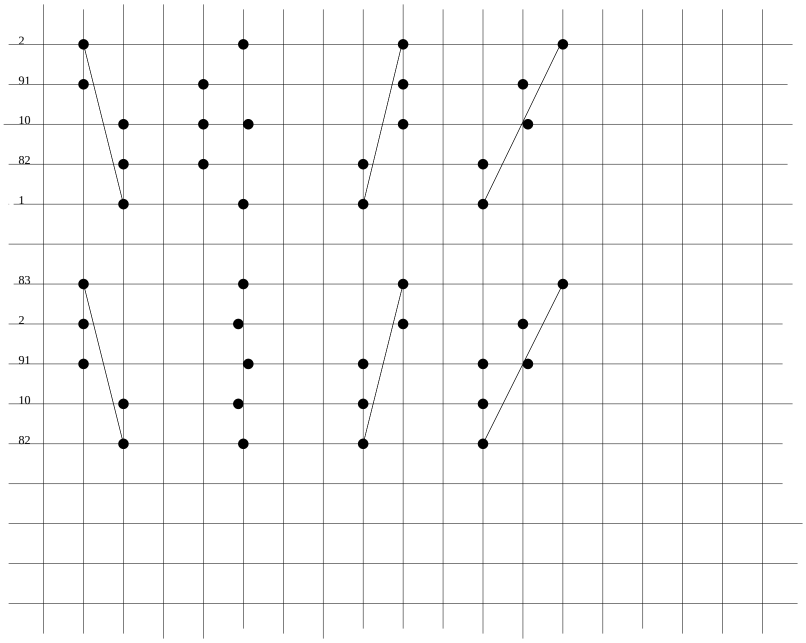

Figure 5: Delta=4

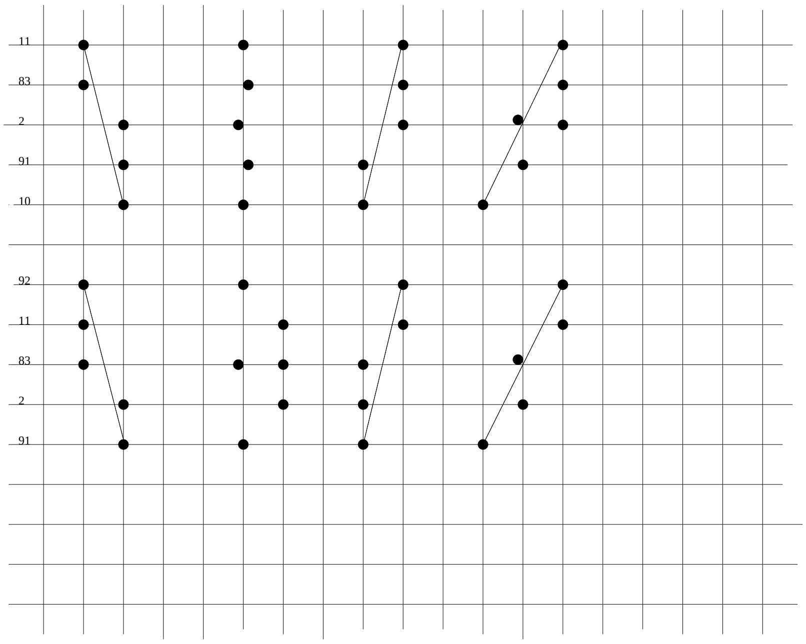

Figure 6: Delta=4 (continued)

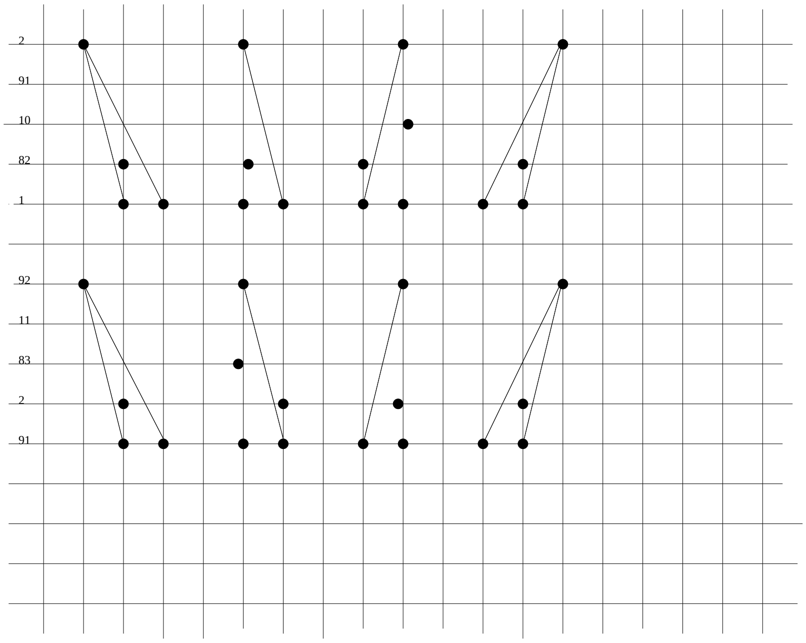

Figure 7: Delta=4 (continued)

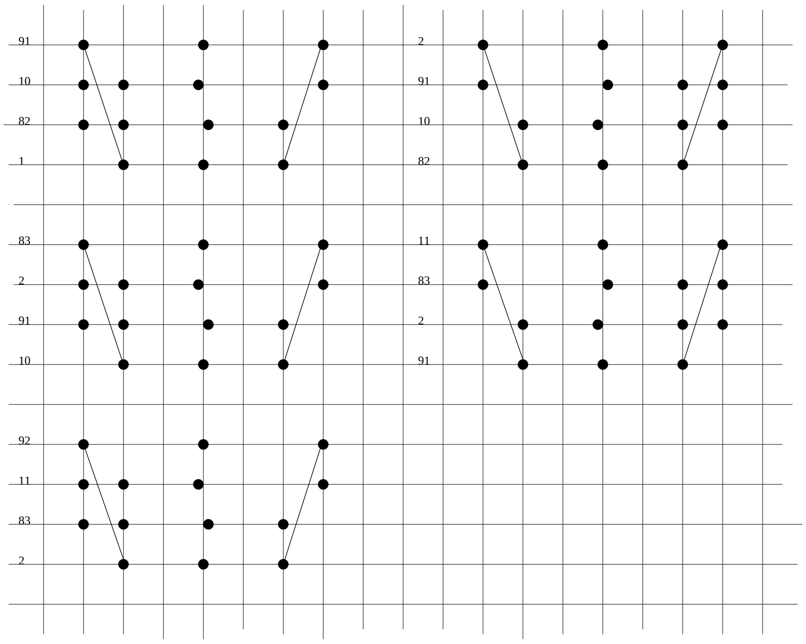

Figure 8: Delta=3

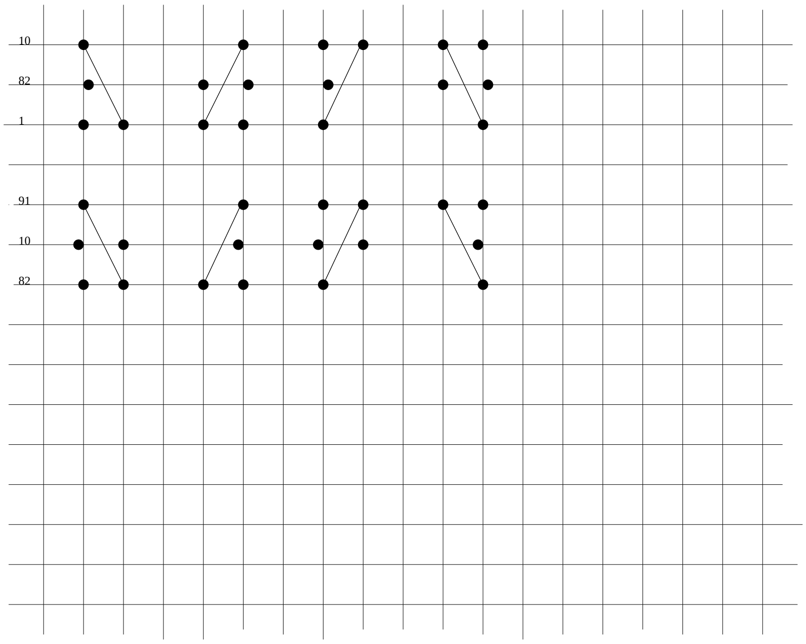

Figure 9: Delta=2

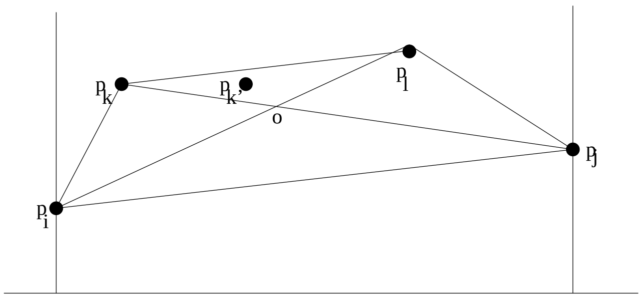

Figure 10: A lower bound on $g_4(n)$