Some Problems on Approximation of Set of Points by Polygonal Curves

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Abstract

Several fundamental problems in Computational Geometry come from the field of Facility Location. In this paper we consider some problems that belong to the interplay between these two areas. Given a set S of points in the plane: (1) find a polygonal chain with one corner belonging to S such that minimizes the maximum distance, (2) find a polygonal chain with k corners belonging to S such that minimizes the maximum distance, (3) find a polygonal chains with constrained length such that minimizes the maximum distance. Many more similar problems are studied in [2]

1 Introduction

The problem of finding polygonal chains which approximate geometric objects has been widely studied by the Computational Geometry community. Its relevance to fields such as geografic information systems, pattern recognition, CAD, etc, as well as its theoretical importance, has originated such an interest. Several authors, Imai and Iri [4], Toussaint [6], Melkman and O'Rourke [5], Chan and Chin [1], among others, have posed and solved many fundamental problems related to that topic.

In those papers the problem of approximating a given polygonal chain by another whose vertices are a subset of the input chain is considered. When the line segment $\overline{p_r p_s}$ approximates the polygonal chain $\{p_r, p_{r+1}, \cdots, p_s\}$, its corresponding error is defined by the maximum Euclidean distance between $p_k, r \leq k \leq s$, and the line segment $\overline{p_r p_s}$. Others distances are also used for defining the error. The error of the approximating chain is the maximum error over all its line segments. Two main problems are considered in the above-mentioned papers, namely: (1) the **min-#** problem: to minimize the number of vertices of an approximating polygonal chain with the error within a given bound, and (2) **min-ϵ** problem: to minimize the error of an approximating polygonal chain consisting of a given number of vertices.

The approximation of sets of points by polygonal chains has arisen in the context of Statistics and Operation Research. Here the vertical distance, also called the Tchebyshev error of the fit, is used to define the error in stead of the Euclidean distance. Hakimi and Schmeichel [3] posed the following problem: Given a set of points S in the plane and an integer $k \geq 0$, construct a polygonal chain P with k or fewer corners (vertices) such that $d(S, P)$ (vertical distance) is as small as possible. We note that the polygonal chain must start at the leftmost point in S and end at the rightmost point. They solve two variants of the problem: when P is required to have its corners on points in S, *the discrete problem*, and when its corners can be on any point in the plane. Those problems can be seen as min-ϵ problems for set of points.

For both of them, Hakimi and Schmeichel give $O(n^2 \log n)$ algorithms which do not work in the presence of degeneracies (they do not admit points with the same x-coordinate). In this paper we present an $O(n \log n)$ algorithm for the 1-corner discrete problem and an $O(n^2)$ algorithm for the k-corner discrete problem that makes no degeneracy assumptions.

In the last section, we consider the problem of constructing a discrete polygonal chain that minimizes the maximum vertical distance to the set of points and whose length is not greater than a certain value.

2 1-Corner Discrete Polygonal Chains

We first observe that the minimum vertical distance to the optimal chain is not necessarily attained at a vertex on the convex hull. The minimum value can be also reached by an interior point, but in that case the point must lie on the vertical line going through the corner of the optimal chain. This property allows us to look for the maximum distance of a polygonal chain among all the vertices of the convex hull and all points which share the x-coordinate of the corner.

For each line segment of a given polygonal chain,

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computing the maximum distance to the vertices of the convex hull can be done in $O(\log n)$ time since the distance function is unimodal. The distance to the interior points that are candidates to reach the maximum can be also computed in $O(\log n)$ time with a suitable preprocessing. We thus state the following theorem.

Theorem 1 *Given a set of n points, finding a 1-corner discrete polygonal chain that minimizes the maximum distance can be done in $O(n \log n)$ time and $O(n)$ space.*

3 k-Corner Discrete Polygonal Chains

For the k-corner discrete problem, let us suppose that we are given the optimal chain. If we divide it into two subchains, at least, one of the two subchains must be optimal for the corresponding subset of points. Otherwise, the chain would not be optimal. This suggests the use of dynamic programming. In particular, each chain of k corners can be decomposed into a 0-corner subchain plus a $(k-1)$-corner subchain. The maximum distances for the former subchains can be obtained in $O(n^2)$ time in a straightforward way; the distances for the latter ones are recursively computed in $O(n^2)$ time and space.

Theorem 2 *Given a set of n points, finding a k-corner discrete polygonal chain that minimizes the maximum distance can be done in $O(n^2)$ time and space.*

4 Polygonal Chains with Constrained Length

In this problem we require that the set $S = \{p_1, p_2, \cdots, p_n\}$ does not contain two points with the same x-coordinate. Let $l_0 > 0$ be the maximum length of the chain and a and b be the leftmost and rightmost points respectively. We first solve a related problem to the original one, namely: given $\lambda > 0$, find a discrete polygonal chain C with minimum length, connecting a and b and such that $d(p_i, C) \leq \lambda$. This problem can be solved by adapting the ideas contained in the paper by Imai and Iri [4]. They construct a directed graph $G(\lambda)$ on which finding the shortest path between a and b gives the solution to the problem above.

We now consider the set $S_{ij} = \{p_i, p_{i+1}, \cdots, p_j\}, i < j$ and define $e_{ij} = max_{i<k<j}\{d(p_k, \overline{p_i p_j})\}$. The minimum of all maximum distances must then belong to the set $\{e_{11}, e_{12}, \cdots, e_{n-1n}\}$. Let us call $\{g_1, g_2, \cdots, g_p\}$ the sorted list in increasing order of the $O(n^2)$ maximum distances for the sake of clarity. A value g_j determines a solution if and only if its corresponding chain has the following two properties: (1) the maximum distance is g_j and its length is not greater than l_0, (2) there exists no other polygonal chain with length not greater than l_0 such that its maximum distance is less than g_j. For each g_j we can solve the problem $min_C\{l(C) \,/\, d(p_i, C) \leq g_j, i = 1, \cdots, n\}$ by using the technique described above. We thus obtain a decreasing sequence of lengths $l_1 \geq l_2 \geq \cdots \geq l_{p-1} \geq l_p$. Using binary search, we can find the first value g_k whose corresponding length is not greater than l_0. We therefore state the following theorem.

Theorem 3 *Given a set of n points, finding a discrete polygonal chain that minimizes the maximum distance and whose length is not greater than a certain value $l_0 > 0$ can be done in $O(n^2 \log n)$ time and $O(n^2)$ space.*

References

[1] Chan W. S., and Chin F., Approximation of Polygonal Curves with minimum number of line segments or minimun error. *International Journal of Computational Geometry & Applications*, Vol. 6, No. 1, 59–77, 1996.

[2] J. M. Díaz, *Localización de estructuras lineales y lineales a trozos*, Ph.D. thesis (in Spanish), Sevilla, mayo, 1998.

[3] Hakimi, S.L. and Schmeichel, E.F., Fitting polygonal functions to a set of points in the plane. *Graphical models and image processing*, Vol. 53, N. 2, 132–136, 1991.

[4] Imai H., and Iri M., Polygonal Approximations of a Curve-Formulations and Algorithms. *Computational Morphology*, editor:G. T. Toussaint, North Holland, 1988.

[5] Melkman A., and O'Rourke J., On Polygonal Chain Approximation. *Computational Morphology*, editor: G. T. Toussaint, North Holland, 1988.

[6] Toussaint G. T., On the Complexity of Approximating Polygonal Curves in the Plane. *Proc. IASTED, International Symposium on Robotics and Automation,* Lugano, Switzerland, 1985.