Towards a contour morphing preserving differentiability

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Abstract

We discuss an approach for the problem of morphing two simple differentiable closed curves composed of quadric arcs maintaining both topology and differentiability in-between. Our approach makes use of a field of directions defined on triangulations.

Introduction

Let $L(i)$, $i = 0, 1$ be two simple differentiable contours consisting of a finite number of quadric arcs. Consider the problem of morphing one into the other preserving both simplicity and differentiability during the transformation. We describe a method for obtaining such a morphing by using systems of trajectories induced by a vector field variable in time. The idea is to extend the method overviewed in [OLIV96] and [KALL97] to preserve simplicity during a morphing of polygons without forcing shape simplifications (e.g., transforming the polygons into their convex hulls) during the process.

Translate $L(1)$ so that it intersects the interior of $L(0)$. Let U_{in} and U_{out} be two polygons and W_{in} and W_{out} two circles, such that $U_{in} \subseteq W_{in} \subseteq Interior(L(0) \cap L(1)) \subseteq W_{out} \subseteq U_{out}$. Also, let T be a triangulation of the annular region U delimited by U_{in} and U_{out} and $V(T)$ be the set of its vertices (see Figure 1). If $D : V \to S_1$ (where S_1 stands for the unit circle) is a discrete field associating each vertex of T with a direction on the plane, then we define $D_T : U \to S_1$ to be the extension of D obtained by linear interpolation within each triangle of T. If D_T is regular (i.e., it contains no singularities), then we say that D is T-regular. Similarly, if D_T is transversal to a contour C in U (i.e., the field along C is nowhere tangent to C), then we say that D is T-transversal to C. We define D to be T-admissible if it is T-regular and all trajectories of D_T start at points in U_{in}, end at points in U_{out} and induce a partial order on the set of triangles in T.[1]

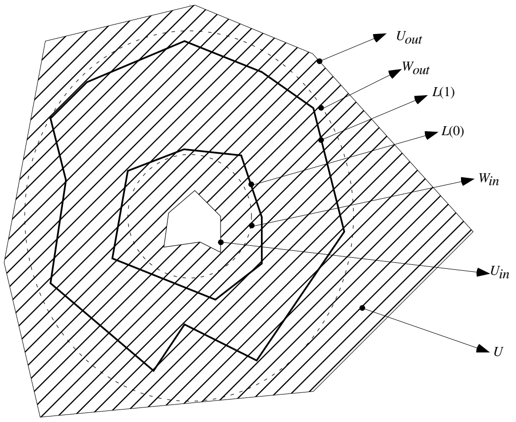

Figure 1: Framework for morphing two curves $L(0)$ and $L(1)$

The proposed method to obtain a morphing of $L(0)$ into $L(1)$ consists of three stages as described below.

The first stage consists of constructing two triangulations[2] of U, named $T(0)$ and $T(1)$, with the same number of vertices. $T(i)$ must be such that there is a discrete field $D(i) : V(T(i)) \to S_1$ which is admissible and transversal to all of W_{in}, W_{out}, $L(0)$, and $L(1)$.

The second stage comprises three different transformations: the set of vertices of $T(0)$ is transformed into the set of vertices of $T(1)$; triangulation $T(0)$ is transformed into $T(1)$ through a sequence of edge flips, and finally, $D(0)$ is morphed into $D(1)$ in such a way that all intermediate fields $D(t)$ are also admissible with respect to their corresponding triangulations $T(t)$.

The third stage consists of finding intermediate instances of the curves being morphed. Each point along any of these curves must belong to a different trajectory induced by $D(t)$. The method chooses a suitable

[1] That order can be defined as follows: If t_0 and t_1 are two triangles in T, $t_0 \leq_D t_1$ iff a trajectory of $\overline{D}$ reaches t_0 first and t_1 after.

[2] Note that a *triangulation* of U may contain vertices which are not on the boundary of U.

set of trajectories and computes an appropriate approximation for each of these. Once this is done, the curves themselves can be obtained by interpolating points taken from each chosen trajectory.

In the remaining sections of this paper we present some key results related with each of these stages.

Computing Triangulations $T(0)$ and $T(1)$

Let L refer to both $L(0)$ and $L(1)$ and let $S \subset U$ be a polygonal neighborhood of L which admits a triangulation T' with n triangles, each one adjacent to exactly two others and intersecting L in a single connected arc. Then, the following result can be applied.

Lemma 1: There is a set V with O(n^2) points, a triangulation T of V and a T-admissible field $D : V \to S_1$ such that D_T is transversal to L.

Although in the worst case n can be arbitrarily larger than the number of quadrics in L, in most cases they are comparable. Pseudo-triangulations where some edges lay on L can be also used to limit n.

Transforming Vertices, Triangulations and Fields

In the first phase of this stage every vertex of $V(0)$ is transported to the position of a corresponding vertex in $V(1)$. Possible ways of obtaining a matching between these two sets of vertices are discussed in the text. Of course, the motion of a vertex can determine changes in the triangulation topology. An elementary change is performed through an edge flip. However, before a flip can occur, the field in both triangles adjacent to the edge being flipped must be modified so that no singularities are introduced. The flip operation itself is not an instantaneous event, but is to be viewed as a continuous process taking finite time. If we consider the quadrilateral in which the edge flip occurs, then the field at any given point p in its interior depends on one set of three vertices just before the flip and on another set of three vertices just after the flip. Therefore, while the flip is being performed the field at p must depend on all four vertices of the quadrilateral.

After all $V(0)$ points are moved to their corresponding positions in $V(1)$,we still have to transform the triangulation until it coincides with $T(1)$. This is made in a second phase, also performed through edge flips. Finally, the field obtained at the end of the second phase is turned into $D(1)$. Essentially, this is done in the following way: Once a vertex $v \in V(1)$ is chosen, one tries to rotate the field at v until it coincides with $D(1)(v)$. However, it is possible that during this rotation the field at v becomes symmetrical to the field at an adjacent vertex w, what would introduce a singularity on the edge $[v, w]$. When this situation is detected, the field at w is forced to rotate at the same velocity as the field at v just before the singularity is introduced. This problem may propagate to a third vertex z adjacent to either v or w, forcing the three vertices to be rotated together. An interesting result is that this process does not cycle. So, the whole process has finite termination as stated in the following lemma.

Lemma 2: Assume that $T(0)$and $T(1)$ have both n vertices and that those on the border of U are the same for both triangulations. Then:

i) There is a sequence ($I_k = [a_k, a_{k+1}]$, $k = 1, \ldots, K$) of $O(n^3)$ closed intervals covering $[0, 1]$ such that $\forall\ t \in I_k\ T(t)$ has the same topology.

ii) Regular fields$D(a_k)$ and $D(a_{k+1})$ can be found and from them $\forall\ t \in I_k$, $D(t)$ can be obtained by an interpolation process which is $O(1)$ for each triangle or quadrilateral.

Note that our appoach does not guarantee the absence of cycles. However, for simple curves, this problem does not usually arise.

Constructing Intermediate Curves

We start with a set of trajectories defined by $D(t)$ and passing through the vertices of $T(t)$. This candidate set may have to be extended whenever an intermediate curve with the desired properties cannot be obtained by a low-order polynomial interpolation process. In this case, a refined triangulation can be obtained following standard approaches (e.g., [AKI70]).

Bibliography

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