Recovering lines with fixed linear probes

Extended Abstract

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1 Introduction

Suppose the only access we have to an arrangement of n input lines is to "probe" the arrangement with horizontal lines. A probe returns the set of *probe points* which are the intersections of the probe's horizontal line (the *probe line*) with all input lines. We assume that none of the input lines is horizontal, so a probe line intersects every input line. Our goal is to reconstruct the set of input lines using a small number of probes.

Aoki, Imai, Imai, and Rappaport [1] observed that if one is allowed to place the probe lines after seeing the results of previous probes, then the number of probes required is at most three.

This paper addresses the problem of *fixed* probes; in our setting the locations of the probes must be chosen before the input is examined. We show that for each natural number n there is a set of $n+1$ probe lines that will serve to determine any arrangement of n input lines and that $n+1$ are sometimes necessary. We further obtain asymptotically tight upper and lower bounds on the maximum number of input lines that are compatible with k probes each of which reports at most n probe points. A line is *compatible* with a set of probes if every intersection of the line with a probe line occurs at a probe point. We give an algorithm to reconstruct an arrangement of n lines that runs in $O(n^2 \log^2 n)$. A randomized version runs in time $O(n^2 \log n)$.

2 Intersection Probes

Let $L \sqcap H = \{\, l \cap h \mid l \in l, h \in H \,\}$. If L is a set of (hidden) non-horizontal lines and H is a set of (known) horizontal probe lines, we want to determine L given just $L \sqcap H$. A natural question is how many probe lines are necessary to determine an arrangement in this way.

Proposition 1 *Let L_1 and L_2 be sets of n non-horizontal lines, and H a set of $n+1$ horizontal lines. If $H \sqcap L_1 = H \sqcap L_2$ then $L_1 = L_2$.*

Proof omitted.

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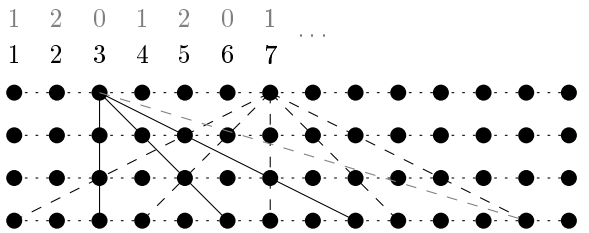

Figure 1: An (n,k) probe set compatible with $n^2/(k-1)$ lines.

Let S be a set of points in $\mathbb{R}^2$. Let $\mathcal{H}(S)$ be the set of horizontal lines that contain a point in S. We say that a set L of non-horizontal lines is *compatible* with S if $S = \mathcal{H}(S) \sqcap L$. We define the *height* of S as $|\mathcal{H}(S)|$. We define the *width* of S as $\max_{l \in \mathcal{H}(S)} |l \sqcap S|$. If S has width n and height k, we call S an (n,k) *probe set*. Let $f_k(n)$ be the maximum cardinality of a set of non-horizontal lines compatible with an (n,k) probe set. Consideration of a rectangular grid of probe points (Figure 1) yields the following.

Proposition 2 $f_k(n) \geq n^2/(k-1) \quad 2 \leq k \leq n$

Proof omitted.

Let $\mathcal{A}(L)$ denote the arrangement induced by a set of lines L. Let $\deg(v)$ be the number of lines that intersect at a vertex v in an arrangement. We will make use in the sequel of the following bound.

Proposition 3 [2] *Let L be a set of N lines and V be any set of M vertices of $\mathcal{A}(L)$.*

$$\sum_{v \in V} \deg(v) \in O(N^{2/3} M^{2/3} + N + M)$$

Our first application of Proposition 3 will be find an asymptotically matching upper bound for the lower bound of Proposition 2.

Theorem 4 $f_k(n) \in O(n^2/k) \qquad 2 \leq k \leq n$

Proof. Consider a set L of N non-horizontal lines and a set H of k horizontal probe lines with at most $n \leq N$ probe points on each; there are $N + k \leq 2N$ lines in total and at most nk points. If all lines are to be compatible with all probes then by Proposition 3 we must have

$$\begin{aligned} Nk \leq \sum \deg(v) &\leq c\left(N^{2/3}(nk)^{2/3} + N + nk\right) \\ N(k-c) &\leq cN^{2/3}(nk)^{2/3} + cnk \\ N^{1/3} &\leq \frac{c(nk)^{2/3}}{k-c} + \frac{cnk}{(k-c)N^{2/3}} \end{aligned}$$

We know that $N \geq n$, so

$$N \in O(n^2/k + n) \qquad \square$$

We now present algorithms to compute L from $L \sqcap H$ for sufficiently large H. In order that the number of input lines need not be known in advance, we assume that the probe points are presented grouped by line. If this is not the case, then an additional $O(nk)$ preprocessing needs to be done for an (n, k) probe set.

Theorem 5 *Let L be a set of n non-horizontal lines and $H = \{h_1, \ldots, h_k\}$ be a set of horizontal lines such that $n < k$. Given $L \sqcap H$ grouped by probe line, we can compute L in time $O(n^2 \log^2 n)$.*

Proof omitted.

Theorem 6 *Let L be a set of n non-horizontal lines and $H = \{h_1, \ldots, h_k\}$ be a set of horizontal lines such that $n < k$. Given $H \sqcap L$ grouped by probe line, we can compute L in expected time $O(n^2 \log n)$.*

Proof. Initially form the set L_0 of $N_0 \leq n^2$ candidate lines obtained by joining every probe point on h_1 with every probe point on h_2. Let $H_0 = H \setminus \{h_1, h_2\}$.

At each step $i \geq 1$, we have a set L_{i-1} of N_{i-1} candidate lines and a set H_{i-1} of at least $n-i$ unprocessed probe lines. Choose a probe line h from the set H_{i-1} at random. Let $H_i = H_{i-1} \setminus \{h\}$. Let L_i be the candidate lines from L_{i-1} that are compatible with h. Forming L_i, that is testing each of the N_{i-1} candidate lines in L_{i-1} to see if it is eliminated, requires $O(N_{i-1} \log n)$ time. By Proposition 1, we know we can stop when $N_i < i$.

We can divide the running of this incremental algorithm into two stages according to whether or not the following is satisfied:

$$(1) \qquad i \leq \frac{n}{2} \qquad \text{and} \qquad N_i > c_1 n$$

for some constant c_1. In stage 2, when (1) is not satisfied, we know from Theorem 4 that $|L_i| \in O(n)$. Both stages together take time at most

$$c_2 \sum_{i=0}^{n/2} N_i \log n + O(n^2 \log n).$$

where the N_i are random variables. We will bound the expected running time of the algorithm by obtaining a bound on $\mathbf{E}N_i$.

Lemma 1 *For $0 \leq i \leq n/2$, $\mathbf{E}N_i \leq c_3 n^{1+(2/3)^i}$*

Proof omitted.

The expected running time $T(n)$ of the algorithm is then bounded as follows

$$T(n) \leq c_2 \sum_{i=0}^{n/2} c_3 n^{1+(2/3)^i} \log n + O(n^2 \log n)$$

We claim that

$$(2) \qquad T(n) \in O(n^2 \log n)$$

To see that (2) holds, note that for $x \geq 8$, $x/2 \geq x^{2/3}$. Thus

$$\sum_{i=0}^{n/2} n^{(2/3)^i} \leq 4n + \sum_{i=0}^{n/2} 2^{-i} n \leq 6n$$

which suffices to prove the claim, and the theorem. $\square$

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References

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