Removing degeneracies by perturbing the problem or perturbing the world*

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Abstract

We describe two problem-specific approaches to remove geometric degeneracies that we call *perturbing the problem* and *perturbing the world*. Using the example of Delaunay triangulation, we show that these approaches lead to relatively simple and efficient perturbations of the points that do not depend on a fixed ordering or index. Extended version at `http://www.inria.fr/RRRT/RR-3316.html`.

1 Introduction

To make the implementation of a geometric algorithm robust, either a significant effort must go into the determining and testing for proper behavior in degenerate cases, or degeneracies must be eliminated. We describe a way to generate problem-specific perturbation schemes that remove degeneracies.

Our approach has three steps: 1) perturb the problem—rather than solving the original in a degenerate case, consider a closely-related problem. 2) perturb the world—extract from the perturbed problem a perturbation of the input, and prove that it resolves all degenerate cases. 3) perturb the code—implement tests that handle degenerate cases consistent with the perturbation. None of these steps are surprising; the same ideas are found in ad-hoc approaches used while coding and in general perturbation schemes. What did surprise us was how many other goals could be met by a problem-specific approach.

In addition to removing degeneracies, an ideal perturbation scheme would have several properties.

1. It should add little computational overhead.
2. It should not require more precision than the test for the existence of a degeneracy.
3. It should be extendable to resolve degeneracies for other types of tests without recomputation.
4. It should lead to output that is valid for an infinitesimally-perturbed input.
5. For testing or verifying randomized or dynamic algorithms, the output should be cannonical irrespective of the order of input.

A number of perturbation schemes [1, 2, 5] use an additional index or increase the computational precision required. We use Delaunay triangulation to show that in low dimensions a problem-specific approach can achieve most of the above simultaneously. Other approaches specifically for the Delaunay complicate the algorithms, primitives [3], or definition [4], and some can produce output that is not the Delaunay triangulation of any set of points [2, 4].

2 2D Delaunay triangulation

For the Euclidean metric, we perturb the problem by perturbing the unit circle. We scale the y coordinates to form a first ellipse; then we shear the x coordinate to create a new ellipse that cannot share more than three points with the original circle and the first ellipse. This removes degeneracies for points that are not collinear, which are the most important cases in implementing a Delaunay triangulation algorithm.

Three or more collinear points are a degenerate case for the Delaunay triangulation under any smooth metric—including our perturbed metric. Since affine transformations preserve collinearity, we also modify the circle in a non-linear fashion. The resulting perturbation of the world sends point (p_x, p_y) to $p' = \big(p_x + \epsilon p_y,\, p_y + \epsilon^2 p_x + \epsilon^3(p_x^2 + p_y^2)\big)$.

Theorem 1 *Let P be a set of distinct points in the plane. Replacing each $p \in P$ with p', as defined above, removes all degeneracies for computing the Delaunay triangulation under the Euclidean metric.*

Proof: With this perturbation, the determinant test for collinearity, $\mathtt{CCW}(p, q, r)$, becomes $D =$

$$(1-\epsilon^3)\begin{vmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{vmatrix} + \epsilon^3 \begin{vmatrix} 1 & p_x & p_x^2+p_y^2 \\ 1 & q_x & q_x^2+q_y^2 \\ 1 & r_x & r_x^2+r_y^2 \end{vmatrix} + \epsilon^4 \begin{vmatrix} 1 & p_y & p_x^2+p_y^2 \\ 1 & q_y & q_x^2+q_y^2 \\ 1 & r_y & r_x^2+r_y^2 \end{vmatrix}$$

If p, q, and r are not collinear, then the sign of D is given by the sign of the first determinant as $\epsilon \to 0$.

When p, q, and r are collinear, the first determinant is zero; we can show that the second and third cannot

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also be zero. Consider lifting the three points to the paraboloid: e.g. $p^* = (p_x, p_y, p_x^2 + p_y^2)$. If p, q, and r are distinct, then $p^*q^*r^*$ define a plane in 3D. If the first determinant is zero, then the plane $p^*q^*r^*$ is vertical; if the second is zero, $p^*q^*r^*$ is parallel to the y-axis; and if the third is zero, $p^*q^*r^*$ is parallel to the x-axis. Since no plane is parallel to all three axes, D cannot be zero unless two points are identical.

An algebraic explanation of the collinear case can also be given. If the line is not vertical, then it has an equation $y = mx + b$, so we can write the second matrix as a Vandermonde matrix whose determinant is zero only if two points have the same x coordinate, which, since they lie on a line that is not parallel to the y-axis, implies that they are identical. When the line is parallel to the y-axis, the second determinant is zero but the third can then be written as a Vandermonde matrix with determinant zero only if two points are identical. Since the determinant of a Vandermonde matrix can be written as a product of differences, its sign is easy to determine. Adding the non-linear terms to the determinant does not necessarily raise the algebraic degree of the test.

Next, consider `InCircle`(p, q, r, s), which tests the sign of determinant

$$D = \begin{vmatrix} 1 & p'_x & p'_y & p' \cdot p' \\ 1 & q'_x & q'_y & q' \cdot q' \\ 1 & r'_x & r'_y & r' \cdot r' \\ 1 & s'_x & s'_y & s' \cdot s' \end{vmatrix}$$

The sign of D locates s with respect to the circle passing through p, q and r. Again, we expand D as a polynomial in ϵ and look at the coefficients of the lowest powers, which determine the sign as $\epsilon \to 0$.

$$\begin{vmatrix} 1 & p_x & p_y & p \cdot p \\ 1 & q_x & q_y & q \cdot q \\ 1 & r_x & r_y & r \cdot r \\ 1 & s_x & s_y & s \cdot s \end{vmatrix} + 2\epsilon(1+\epsilon) \begin{vmatrix} 1 & p_x & p_y & p_x p_y \\ 1 & q_x & q_y & q_x q_y \\ 1 & r_x & r_y & r_x r_y \\ 1 & s_x & s_y & s_x s_y \end{vmatrix} + \epsilon^2 \begin{vmatrix} 1 & p_x & p_y & p_y^2 \\ 1 & q_x & q_y & q_y^2 \\ 1 & r_x & r_y & r_y^2 \\ 1 & s_x & s_y & s_y^2 \end{vmatrix} + \epsilon^3 D',$$

where D' is a polynomial in ϵ that we expand later if these three determinants are zero.

If all three determinants are zero, we are able to conclude that any conic passing through p, q and r must also include s. Suppose s does not lie on the line through p and q, however. Then there is a degenerate conic consisting of the line $\overline{pq}$ and any line through r that does not include s. Thus, either the above determinants are not all equal to zero, or the four points are collinear.

From an algorithmic point of view, we are done. The main use of the `InCircle` test is to locate a query point with respect to the circle circumscribing a triangle of the Delaunay triangulation of a set of points. The only triangles with collinear points, therefore, are those on the convex hull of the point set, and if a bounding triangle or box (or point at infinity) is used to begin the triangulation, then the `InCircle` test cannot be called with four collinear points. Even so, expansion of D' shows that the degeneracy is also resolved in the collinear case.

Notice that the test on perturbed points has the same degree as the original. In fact, the `InCircle` test is usually evaluated by minors of the last column. In the case of co-circular but not collinear points, this expansion is reused, changing the last column to the square of ordinates and, if necessary, to the product of abscissae and ordinates. The collinear case is detected when the minors are all zero, and the degeneracy is resolved by three comparisons of point coordinates. ∎

3 Application to other problems

This approach is even easier to apply to the Delaunay triangulation defined by L_∞ and polygonal metrics, because only affine transformations of the metric are necessary. See the full paper for more detail.

In three dimensions it is helpful to construct a perturbation in stages; using low-order perturbation terms to resolve low order degeneracies, and then adding on higher order terms as necessary. This can also be a useful technique if there are only certain degeneracies that need to be handled by an algorithm, such as in arrangements of segments, where one can choose whether segments sharing endpoints, or having endpoints on the same vertical line, should be regarded as normal or as a degeneracy.

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