Perfect Divisions of a Cake

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The problem studied in this paper arises from a simple practical problem: how to divide a cake among the children attending a birthday party in such a way that each child gets the same amount of cake and (perhaps more important to them) the same amount of icing.

Let S be a convex set contained in the (x, y)-plane. In mathematical terms, a cake C with base S is a solid containing all the points with coordinates (x, y, z) such that $(x, y, 0) \in S$ and $0 \leq z \leq h$, $h > 0$; h is called the height of C. The exposed area of C consists of the boundary of C minus S, i.e. the base of a cake is not considered to be exposed. A cake will be called a *polygonal cake* if S is a convex polygon.

A division of a cake C into k parts by a series of vertical cuts is said to be *perfect* if each part is convex and has the same volume and the same exposed area of S.

Our birthday cake problem can be stated as follows: given a cake C, does it have a perfect partitioning into k pieces? If a cake has such a partitioning, we will also say that C can be cut *perfectly*.

A cake whose base is a square can be cut perfectly into three pieces as follows: take any three points x, y and z that divide the perimeter of its base into three pieces of the same length. Now make vertical cuts along the line segments connecting these points to the center of the base of the cake; see Figure 1.

Perfect partitionings of cakes in which the vertical cuts are all along line segments concurrent at a point p are called *radial perfect partitionings*.

Notice that for any $k > 0$, any circular cake C has a radial perfect partitioning into k pieces. This motivates the following definition.

A cake C is called *graceful* if, for every k, there is a perfect radial partitioning of C into k pieces. A natural question arises here: is it true that a graceful cake must necessarily be circular? We will prove that the answer to this question

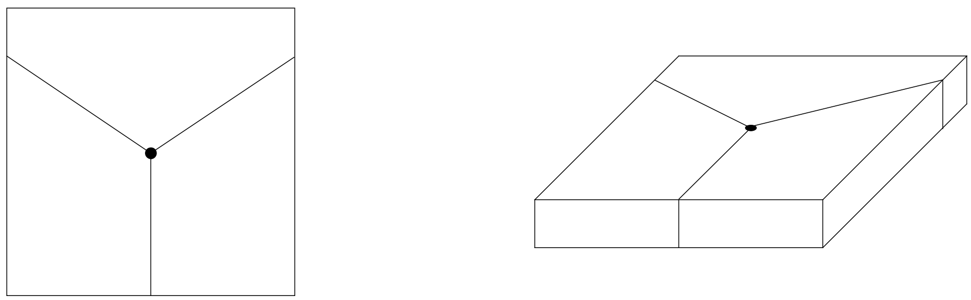

Figure 1: Cutting a square cake into 3 pieces.

is “no”. We will show that there are an infinite number of graceful polygonal cakes, and give a full characterization of them. We will exhibit polygonal cakes that cannot be cut perfectly into four pieces by radial cuts.

There are perfect partitionings of rectangular cakes that are not radial. A non-radial perfect partitioning of a cake whose face is a rectangle of size 2 by 4 can be obtained by making vertical cuts along the line segments that divide its base into four parts each with equal perimeter, as shown in Figure 2.

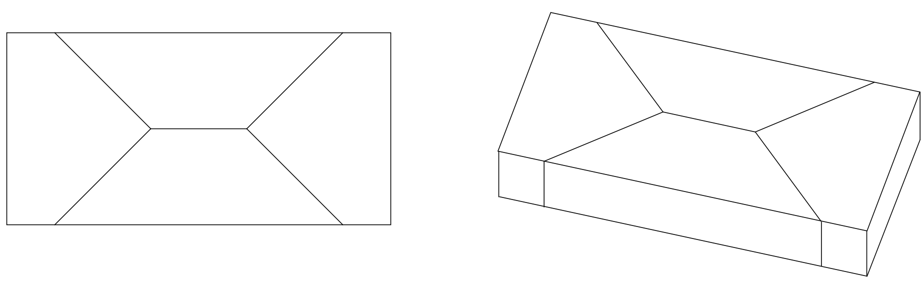

Figure 2: A non-radial perfect partitioning of a rectangular cake into four pieces.

We will conclude by showing that any cake can be perfectly partitioned into k pieces for all $k > 0$. Of course these partitionings are not necessarily radial.