A shrinking-circle sweep-algorithm to construct the farthest site Delaunay diagram in the plane

– Extended abstract –

B. Adam, P. Kauffmann, D. Schmitt, and J. -C. Spehner *

Abstract

We present a new way to compute the farthest site Delaunay diagram of a planar set S of n sites in $O(n \log n)$ time by using a sweep technique. We sweep the plane by a circle whose center is a fixed point in the convex hull of S and whose radius decreases from $+\infty$ to 0.

1 Introduction

The *farthest site Delaunay diagram* $Del_{-1}(S)$ is dual to the *farthest site Voronoi diagram* $Vor_{-1}(S)$. Different kinds of algorithms were given to construct these diagrams in the plane ([AGSS89], [Sky91], ...). Sweeping the plane is a well known method to construct classical Voronoi and Delaunay diagrams ([For87], ...). In this paper we present a sweep algorithm to construct $Del_{-1}(S)$.

The plane is swept by a circle $\mathcal{C}$ that first contains all the sites of S in its interior and whose radius decreases. The center O of $\mathcal{C}$ is a fixed point in the convex hull $conv(S)$ of S. As $\mathcal{C}$ shrinks, vertices, edges, and regions are inserted in the current diagram.

2 The definition of $Del_{-1}(S)$

Let E be the Euclidean plane and S be a planar set of n sites of E.

A circle σ_{-1} of E whose interior contains a subset P of S is said to be *S–anti–critical*, if the remaining sites of $Q = S \setminus P$ are on σ_{-1}; Q is then called an *anti–section*.

$Del_{-1}(S)$ is a partition of E whose *vertices* are the vertices of $conv(S)$. For every anti–section $\{s, t\}$, the open straight line segment $h_{-1}(s, t)$ linking s to t is an *edge* of $Del_{-1}(S)$. For every anti–section Q such that $|Q| > 2$, the interior $h_{-1}(Q)$ of $conv(Q)$ is a *region* of $Del_{-1}(S)$. If $|Q| > 3$, $h_{-1}(Q)$ is not a triangle and $Del_{-1}(S)$ is not a triangulation.

*Laboratoire MAGE, Université de Haute-Alsace, 68093 Mulhouse, France
[B.Adam|P.Kauffmann|D.Schmitt|JC.Spehner]@univ-mulhouse.fr

3 The validated diagram and the events

From this point we assume that all the sites of S lie on the boundary $\delta(conv(S))$ of $conv(S)$.

Given a radius ρ of the shrinking sweep-circle $\mathcal{C}$ centered at O, let T be the sites that have already been swept by $\mathcal{C}$.

A *site* s [resp. an *edge* $h_{-1}(s, t)$, resp. a *region* $h_{-1}(Q)$] is said to be *validated relatively* to (T, ρ) if there exists a closed disk ω such that $T \cup \mathcal{C} \subset \omega$ and $\delta(\omega) \cap S = \{s\}$ [resp. $\delta(\omega) \cap S = \{s, t\}$, resp. $\delta(\omega) \cap S = Q$]. The subdiagram of $Del_{-1}(S)$ composed of the validated vertices, edges, and regions is denoted by $Valid(T, \rho)$. During the sweep, $Valid(T, \rho)$ is modified by adjunctions of validated vertices, edges, and regions which are called the *events* of the sweep.

The sweeping of a site s is an event if and only if $\mathcal{C}$ and T are on the same side of the tangent to $\mathcal{C}$ at s. Indeed, there exists a circle tangent to $\mathcal{C}$ at s, passing through a site q of T and whose interior contains $T \cup \mathcal{C}$. Therefore s and $h_{-1}(s, q)$ are validated.

Otherwise, there exists an unique neighbour q of s on $\delta(conv(S))$ that is not on the same side of the tangent to $\mathcal{C}$ at s as O. Thus s belongs to the set $vis(q)$ of points of $\mathcal{C}$ that are visible from q (i.e., that are not hidden by $\mathcal{C}$). When $\mathcal{C}$ will be tangent to the straight line passing throught s and q, s and $h_{-1}(s, q)$ are validated. This tangent point is called the *limit point* of sq denoted by $lim(s, q)$ and its sweeping is an event.

The validation of a region $h_{-1}(Q)$ happens when $\mathcal{C}$ becomes tangent to the circumcircle circ(Q) of Q such that the interior of circ(Q) contains $\mathcal{C}$. This tangent point is called the *ultimate point* of Q and denoted by $ult(Q)$. After its sweeping, all the edges of $h_{-1}(Q)$ are validated.

Proposition 1 *$Valid(T, \rho)$ is connected and the events described above are the only events of the sweep.*

Proof. The validation of a site other than the first one connects this site to a validated site of $Valid(T, \rho)$. Thus, $Valid(T, \rho)$ is connected. Since $\mathcal{C}$ sweeps the whole plane all the ultimate points will be swept and all $Del_{-1}(S)$ regions and there boundaries will be validated. Moreover, when an edge that connects two validated vertices is validated, a region is also validated. Thus, the events above are the only events of the sweep.□

4 The location of a point on $\mathcal{C}$

When $\mathcal{C}$ decreases and sweeps a site s, we search a site q in T such that s and $h_{-1}(s,q)$ can be validated. Since $Valid(T,\rho)$ is planar and connected, q can only be a site on $\delta(Valid(T,\rho))$. Therefore, we define a partition of $\mathcal{C}$ associated to this boundary.

Every oriented edge st of $\delta(Valid(T,\rho))$ such that st admits some points of $\mathcal{C}$ on its left, is called a *frontal edge*. The sequence of these edges is called the *Delaunay front* and is denoted by $\mathcal{F}_{-1}$. For every couple (pq, qr) of consecutive edges of $\delta(Valid(T,\rho))$, the *triple* (p,q,r) is said to be *frontal* if either pq or qr is frontal.

Lemma 1 *If $vis(T) = \cup\{vis(q); q \in T\}$, for every point s of $\mathcal{C} \setminus vis(T)$,*
(i) either there exists an unique frontal edge pq such that the circle $C(p,q)$ passing through p and q tangent to $\mathcal{C}$ at a point $pat(p,q)$ on the left of pq is S–anti–critical. $pat(p,q)$ is said to be attached to pq.
(ii) or there exits an unique frontal triple (p,q,r) such that the circle $D(q,s)$ passing through q tangent to $\mathcal{C}$ at s is S–anti–critical. The set of points of $\mathcal{C} \setminus vis(T)$ attached to (p,q,r) is denoted by $arcat(p,q,r)$.
This mapping from $\mathcal{F}_{-1}$ to the partition of $\mathcal{C} \setminus vis(T)$ is one–to–one.

To locate a point s in this partition, we use a balanced binary search tree such that each node is a frontal edge. The frontal edges are sorted according to their order in $\mathcal{F}_{-1}$. Thus we can locate a point s of $\mathcal{C} \setminus vis(T)$ in logarithmic time in the number of frontal edges.

5 The algorithm

5.1 The site event

From lemma 1, when $\mathcal{C}$ sweeps over a site s of $\mathcal{C} \setminus vis(T)$, s is validated. Moreover, either $s \in arcat(p,q,r)$ and $h_{-1}(s,q)$ is validated, or $s = pat(p,q)$ and $h_{-1}(s,p)$, $h_{-1}(s,q)$, and $h_{-1}(s,p,q)$ are validated.
From section 3, when $s \in vis(T)$ there exists a neighbour site q of s such that $s \in vis(q)$. Therefore, the limit point $lim(s,q)$ is generated and inserted in a binary search tree according to its distance to O.

5.2 The limit point event

When $\mathcal{C}$ sweeps $lim(s,q)$, s and $h_{-1}(s,q)$ are validated.

5.3 The ultimate point event

From section 3, when $\mathcal{C}$ sweeps $ult(s_i,...,s_k)$, the region $h_{-1}(s_i,...,s_k)$ and its only non yet validated edge $h_{-1}(s_i,s_k)$ are validated.

5.4 Updating $\mathcal{F}_{-1}$

$\mathcal{F}_{-1}$ has to be updated when a frontal edge is either created or deleted (i.e., when an event occurs, and when $\mathcal{C}$ becomes tangent to a frontal edge).

An ultimate point is generated when a new frontal triple (p,q,r) such that r is on the left of pq appears in $\mathcal{F}_{-1}$. This ultimate point is inserted in a binary search tree, and will be deleted when (p,q,r) is no longer frontal.

6 The algorithm complexity

Lemma 2 *If V is the set of validated sites of T,*
(i) the number k of frontal edges verifies $k \leq 2|V| - 2$;
(ii) the number of living ultimate points is less than or equal to k;
(iii) the number l of living limit points verifies $l \leq 2|V|$.

Theorem 1 *If S is a planar set of n sites, the shrinking-circle algorithm computes $Del_{-1}(S)$ in $O(n \log n)$ time.*

Proof. The preliminary sorting of S is done in $O(n \log n)$ time. Since $Del_{-1}(S)$ is a partition of a convex polygon with n vertices, $Del_{-1}(S)$ admits at most $2n-1$ edges. Thus the algorithm handles $O(n)$ events. Using lemma 2, every search, insertion, or deletion in the binary trees is done in $O(\log n)$ time. Therefore the construction of $Del_{-1}(S)$ is done in $O(n \log n)$ time. Moreover, $Vor_{-1}(S)$ can be deduced in $O(n)$ time.□

7 Conclusion

Our circle sweep–algorithm is interesting notably when S is not supposed to be a convex set. Indeed, with little modifications, it computes simultaneously $Del_{-1}(S)$ and $conv(S)$. Since anti–critical circles and anti–sections fully define both $Del_{-1}(S)$ and $Vor_{-1}(S)$, this algorithm can compute directly $Vor_{-1}(S)$. In this case, the validated farthest site Voronoi diagram is defined as follows: for every $arcat(p,q,r)$, the set of centers of the circles $D(q,s)$ is an hyperbolic arc. The restriction of $Vor_{-1}(S)$ that is inside the union $\mathcal{H}$ of all the hyperbolic arcs is validated. $\mathcal{H}$ corresponds to the parabolic or elliptic fronts of the sweep algorithms given in [For87] and [AKSS97].

References

[AGSS89] A. Aggarwal, L. J. Guibas, J. Saxe, and P. W. Shor. A linear-time algorithm for computing the Voronoi diagram of a convex polygon. *Discrete Comput. Geom.*, 4(6):591–604, 1989.

[AKSS97] B. Adam, P. Kauffmann, D. Schmitt, and J.-C. Spehner. An increasing–circle sweep–algorithm to construct the Delaunay diagram in the plane. In *Proc. 9th Canad. Conf. Comput. Geom.*, pages 268–273, 1997.

[For87] S. J. Fortune. A sweepline algorithm for Voronoi diagrams. *Algorithmica*, 2:153–174, 1987.

[Sky91] S. Skyum. A simple algorithm for computing the smallest enclosing circle. *Inform. Process. Lett.*, 37:121–125, 1991.