On the $\Omega(n^{4/3})$ Weak Lower Bounds for Some 3D Geometric Problems *

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Abstract

In this paper, we establish the $\Omega(n^{4/3})$ lower bounds constructively for some 3D geometric problems. The model we are using is essentially that of [ES93]. However, besides the primitive which can be used to solve a specific problem directly (with a brute-force method) we also allow the primitive "Is this point to the left or right of (above or below) the intersection of these two lines?", which has been used to divide some similar problems into subproblems and obtain good upper bounds. Our problems include computing the shortest red/blue vertical distance between a set of red and blue line segments in 3D and computing the depth order of a set of line segments in 3D, etc.

1 Introduction

Proving the lower bounds of problems is one of the central part in algorithm theory. Established lower bound for a specific problem usually convinces people not to try obtaining better algorithms unless under a different model of computation or when some extra primitives are allowed. In computational geometry (as well as in the general algorithm design area) lower bound is usually obtained via problem reduction, probablistic argument, combinatorial counting. However, in general the lower bound results are very sparse compared with the vast upper bound results in computational geometry [Cha94].

We briefly mention three techniques for proving lower bounds in computational geometry. The most fundamental technique is problem reduction. Many fundamental problems in two and three dimensions, like convex hull, Voronoi Diagram, Euclidean Minimum Spanning Tree, closest pair, etc., can be shown to have $\Omega(n \log n)$ lower bounds by a reduction from sorting, element uniqueness, set disjointness, etc [PS85]. Similarly, most of the NP-hard results in computational geometry are obtained via problem reduction. Another important technique is combinatorial counting, i.e., computing the combinatorial complexity of the problem. For instance, we can simply obtain the $\Omega(n^2)$ lower bound for constructing the arrangement of n lines in the plane. A nontrivial technique in this respect is to use the Davenport-Schinzel sequence [DS65, HS86] to establish the lower bound of (the space complexity of) some geometric problems. An example is the $\Omega(n\alpha(n))$ lower bound for constructing the lower envelope of a set of n line segments. The probablistic argument method is largely used to prove the lower bounds of range queries, simplex queries, etc [Cha89, Cha90a, Cha90b, CR92].

Besides the above standard lower bound techniques, Gajentaan and Overmars recently defined a class of 3SUM-hard problems [GO93]. They showed that unless the 3SUM problem can be solved in $o(n^2)$ time there is no hope to solve this class of geometric problems in $o(n^2)$ time. Erickson and Seidel proved, with an adversary argument, that the problem of testing whether a set of points in d-dimension is *degenerate* requires $\Omega(n^d)$ sideness queries [ES93]. The model of Erickson and Seidel, which will be called *ES-model* throughout this paper, restricts any algorithm to have only a fixed primitive which "seems" of direct use for solving a specific problem. (Usually such a primitive enables us to solve the problem with an easy brute-forth algorithm.) For the problem of testing whether a set of points in d-dimension is *degenerate*, it is clear that such a primitive is the *sideness query*, i.e., testing whether $d+1$ points are coplanar on a d-hyperplane [ES93].

In the field of computational geometry there are a list of problems whose best known upper bounds are a little bit higher than $O(n^{4/3})$ and most of these

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results are obtained via randomized divide and conquer. A typical example is the famous Hopcroft's problem: Given n points and n lines in the plane, does any point lie on any line? Another example, which has application in computer graphics, is the following: Given n line segments (lines) in space, compute the depth order of these line segments (lines). The best $O(n^{4/3}2^{O(\log^* n)})$ upper bound for Hopcroft's problem is obtained in [Mat93]. The best $O(n^{4/3+\epsilon})$ upper bound for computing the depth order of a set of lines is due to Chazelle et al. [CEG+90]. The same upper bound for computing the depth order of a set of line segments is due to de Berg et al. [dBOS92]. Although great effort has been made, no upper bound better than $O(n^{4/3})$ has been achieved. Consequently, it is meaningful to study the lower bounds for these problems.

As with Hopcroft's problem, Erickson recently established an $\Omega(n^{4/3})$ lower bound for all the partitioning algorithms solving this problem [Eri95a]. With respect to the ES-model, if the only allowed primitive is "On which side of this line does this point lie?" then it is easy to prove the $\Omega(n^2)$ lower bound for Hopcroft's problem. Since Hopcroft's problem has already been solved in $o(n^2)$ time this implies that the ES-model is too weak (for most of these geometric problems). However, if either of the primitives "Is this point to the left or right of (above or below) the intersection of these two lines?" and "Is the slope of this line larger or smaller than the slope of the line connecting these two points", which have been used to obtain some of those $o(n^2)$ randomized divide and conquer algorithms, is also allowed then it is not known whether nontrivial lower bounds can be obtained on the ES-model.

In this paper, we establish the $\Omega(n^{4/3})$ weak lower bounds [1] for several geometric problems which are closely related to Hopcroft's problem. An enhanced ES-model is used throughout this paper, i.e., besides the primitives seemingly of direct use for these problems the primitive "Is this point to the left or right of (above or below) the intersection of these two line segments?" is allowed. (Since this model is not a general one, we use the term "weak" to distinguish the lower bounds under this model from the standard lower bounds under the algebraic computation tree model. For the ease of presentation we mix the use of "lower bound" and "weak lower bound" and from now on any lower bound mentioned will be under the above enhanced ES-model, unless otherwise specified.) The four problems we consider are as follows.

[1] The best known lower bound for these problems under the algebraic computation tree model is $\Omega(n \log n)$.

(1) Given n red line segments and n blue line segments in 2D, decide whether there is a red/blue intersection.

(2) Given n red line segments and n blue line segments in 3D, compute the shortest red/blue vertical distance between them.

(3) Given n red line segments and n blue line segments in 3D, decide whether all the red segments are above the blue segments.

(4) Given n line segments 3D, compute the depth order of these segments.

The first and the third problem are special cases of the second problem. Therefore we will mainly study the second and the fourth problem. As with the second problem, it can clearly be solved in $O(n^2)$ time with a brute-force method. The primitive of such a brute-force algorithm is the *red/blue vertical distance*: given a pair of red and blue line segments in 3D, return the vertical distance between them. We prove that with the red/blue vertical distance function, together with the primitive "Is this point to the left or right of (above or below) the intersection of these two line segments?" $\Omega(n^{4/3})$ is the lower bound for this problem. The technique used is mainly combinatorial counting and an adversary argument similar to [ES93]. The lower bound of the fourth problem can be established similarly with a more complex proof.

2 Preliminaries

In this section we present some known results which are the basis for our lower bound results in the subsequent sections. We first introduce the following lemma.

Lemma 1. Let $\phi(j)$ denote Euler's function: $\phi(j) = |\{i|1 \le i \le j \text{ and } gcd(i,j) = 1\}|$. Then

(1) $\sum_{j \le m} \phi(j) = \frac{3}{\pi^2}m^2 + O(m \log m)$,

and

(2) $\sum_{j \le m} j\phi(j) = \frac{3}{\pi^2}m^3 + O(m^2)$.

This lemma is proved in [HW65] (page 268) and is used in [Fre81]. Suppose we have a set of m^2 points defined by $M = \{(i,j)|1 \le i,j \le m\}$. Given a line l in the Euclidean plane, we refer to the number of points of M through which l passes as the *rank* of l with respect to M. The following lemma is proved in [Fre81] and since it is crucial to our result we also rewrite the proof.

Lemma 2 [Fre81]. There exists $O(m^2)$ distinct lines in the Euclidean plane so that their sum of ranks with respect to M is $\Omega(m^{8/3})$.

Proof. Let $l(i,j,a,b)$ denote the line passing through the points (i,j) and $(i+a, j+b)$. First we define a class of lines $F_m = \{l(i,j,a,b) | 1 \le a \le \lfloor m^{1/3} \rfloor, 1 \le i \le a, 1 \le j \le m/2, 1 \le b \le a$, and $gcd(a,b) = 1\}$. Second we show that all lines in F_m are distinct. Suppose that $l(i,j,a,b) = l(i',j',a',b')$. Since $l(i,j,a,b) = l(i',j',a',b')$, $b/a = b'/a'$ and consequently; since $gcd(a,b) = 1$, $gcd(a',b') = 1$ we have $a = a', b = b'$. Since $l(i',j',a',b')$ passes through (i,j) and $gcd(a',b') = 1$, $i' = i(\text{MOD}a)$, and therefore; since $1 \le i, i' \le a$, it follows that $i = i'$. Similarly we have $j = j'$.

The number of lines in F_m is given by $\lfloor m/2 \rfloor \sum_{a=1}^{\lfloor m^{1/3} \rfloor} a\phi(a) \le m^2/\pi^2 + O(m^{5/3}) \le m^2$ (when m is sufficiently large). The rank of $l(i,j,a,b)$ with respect to M is at least $\text{MIN}(m/a, m/2b) \ge m/2a$. Therefore the sum of ranks of the lines in F_m with respect to M, is at least $(m/2a) \times \lfloor m/2 \rfloor \sum_{a=1}^{\lfloor m^{1/3} \rfloor} a\phi(a)$ $= (m/2)\lfloor m/2 \rfloor \sum_{a=1}^{\lfloor m^{1/3} \rfloor} \phi(a) = \Omega(m^{8/3})$. □

In the subsequent sections, we establish our lower bound results based on Lemma 2.

3 Weak lower bound for the shortest red/blue vertical distance problem

The vertical distance between a pair of red and blue segments in 3D is defined as the length of the vertical line segment connecting the two segments and if such a vertical connecting line segment does not exist then the distance is defined as $+\infty$. The problem of computing the shortest red/blue vertical distance between a set of n red segments and a set of n blue segments can clearly be solved in $O(n^2)$ time with a brute-force method. The primitive of such a brute-force algorithm is the *red/blue vertical distance (RBVD)*: given a pair of red and blue line segments in 3D, return the vertical distance between them. If the only primitive allowed is RBVD then it is easy to prove that $\Omega(n^2)$ is the lower bound for this problem. We show that if the primitive "Is this point to the left or right of (above or below) the intersection of these two line segments?" is also allowed then $\Omega(n^{4/3})$ is the lower bound of this problem.

Theorem 3. There exist $O(n)$ distinct red and blue line segments in the Euclidean space so that it takes $\Omega(n^{4/3})$ time to compute the shortest red/blue vertical distance between these line segments even if the primitive "Is this point to the left or right of (above or below) the intersection of these two line segments?", besides RBVD, is allowed.

Proof. Let $s(x, y, x_1, y_1)$ be a line segment connecting (x,y) and $(x+x_1, y+y_1)$ in the Euclidean plane. We construct a set of 2D line segments $G_m = \{s(i+\alpha_{ij}, j+\beta_{ij}, \alpha_{ij}, 0) | 1 \le i,j \le m \text{ and } \alpha_{ij}, \beta_{ij} > 0\}$. In other words, the line segments in G_m are all horizontal, tiny and are at the upper right corner of the points in M. α_{ij}, β_{ij} can be made very very small. It is clear that $|G_m| = m^2$.

The set of 3D blue line segments $\mathcal{F}_m$ is obtained by taking an arbitrary set of 3D lines whose vertical projections corresponding the set of lines in F_m. For each point (i,j) in M we associate a set $S(i,j)$ which contains all the lines in $\mathcal{F}_m$ whose planar projection pass through (i,j). Clearly, following Lemma 2, $\sum_{i,j} |S(i,j)| \ge m^{8/3}$.

We construct a set of 3D red line segments $\mathcal{G}_m = \{s_{ij} = [s(i+\alpha_{ij}, j+\beta_{ij}, \alpha_{ij}, 0), Z = z_{ij}] | 1 \le i,j \le m$, and $\alpha_{ij}, \beta_{ij}, z_{ij} > 0\}$. In other words, the line segments in $\mathcal{G}_m$ are obtained by elevating the segments in G_m vertically to 3D.

Now the adversary controls the red line segments $\mathcal{G}_m$ by selecting suitable α_{ij}, β_{ij}, for all i,j, so that the shortest vertical distance between the red/blue segments is infinity. To report this shortest vertical distance, any algorithm must compute, for all i,j, the vertical distance between all the line segments in $S(i,j)$ and s_{ij}. If an algorithm fails to check the distance between some segments in $S(i,j)$ and s_{ij}, the adversary perturbs α_{ij}, β_{ij} so that the distance between some unchecked line segments in $S(i,j)$ and s_{ij} becomes the minimum (i.e., finite). This holds for all $S(i,j)$ since s_{ij} (consequently α_{ij}, β_{ij}) are independent with each other for all i,j, and moreover, after this perturbation the algorithm can not distinguish the two input with the RBVD function and the primitive "Is this point to the left or right of (above or below) the intersection of these two line segments?" [2]. Consequently, computing the shortest vertical distance between the blue segments in $\mathcal{F}_m$ and the red segments in $\mathcal{G}_m$ takes at least $\sum_{i,j} |S(i,j)| \ge m^{8/3}$

[2]However, if the primitive "Is the slope of this line larger or smaller than the slope of the line connecting these two points" is allowed then the algorithm can detect the perturbation and therefore this proof fails.

time.

Finally, to complete the proof of the theorem we simply set $n = m^2$ and make the lines in $\mathcal{F}_m$ into line segments. □

We can apply the above proof immediately to establish the lower bound for detecting the red/blue intersection among a set of red segments and a set of blue segments in 2D.

4 Weak lower bound for testing the towering property of red/blue segments

It should be note that the above proof can be directly used to prove the lower bound of testing the "towering property of red/blue segments": given n red segments and n blue segments, does there exist a red segment above some blue segments? Clearly the primitive of direct use to this problem is the *red/blue aboveness testing function*: given a pair of red and blue segments, it returns YES if the red segment is above the blue one, it returns NO if the red segment is below the blue one and it returns NIL if the vertical projections of the two segments have no intersection (or equivalently, if the vertical distance of the two segments is $+\infty$). We simply state the result as follows.

Corollary 4. There exist $O(n)$ distinct red and blue line segments in the Euclidean space so that it takes $\Omega(n^{4/3})$ time to decide whether there exists a red segment above some blue segments even if the primitive "Is this point to the left or right of (above or below) the intersection of these two line segments?", besides the red/blue aboveness testing function, is allowed.

A closely related problem of testing the "towering property of red/blue lines": given n red lines and n blue lines, does there exist a red line above some blue lines? (or equivalently, are all blue lines above all red lines?) can be solved in $O(n^{4/3+\epsilon})$ time [CEGS89]. This problem has been used to obtain the $O(n^{4/3+\epsilon})$ upper bounds for computing the shortest vertical distance between two terrains [CEGS89], computing the depth order of n lines [CEG+90] and computing the longest vertical distance between n lines [GP92]. However, the above lower bound result on testing the towering property of red/blue segments can not be claimed on lines.

5 Weak lower bound for computing the depth order of a set of line segments

In this section, we prove the lower bound for computing the depth order of a set of n line segments in 3D. A depth order of a set of 3D line segments is an order such that segment s_1 comes before s_2 if s_2 is above s_1. It is easy to see that a depth order of a set of segments in 3D does not always exist since there can be cyclic overlap among segments. The best known upper bound for computing the depth order of a set of line segments is due to de Berg et al. [dBOS92].

Clearly, the *aboveness testing function* is the primitive which can be applied to solve this problem directly. (The *aboveness testing function*: given a pair of 3D segments (rods) s_1 and s_2, it returns YES if s_1 is above s_2, it returns NO if s_1 is below s_2 and it returns NIL if the vertical projections of s_1, s_2 have no intersection (or equivalently, if the vertical distance of the two segments is $+\infty$).) We prove that the lower bound of this problem is $\Omega(n^{4/3})$ even an extra primitive "Is this point to the left or right of (above or below) the intersection of these two line segments?" is allowed. In fact we prove a stronger result that even checking whether the n rods have a cyclic overlap takes $\Omega(n^{4/3})$ time with these two primitives.

Theorem 5. There exist $O(n)$ distinct line segments in the Euclidean space so that it takes $\Omega(n^{4/3})$ time to compute the depth order of these segments even if the primitive "Is this point to the left or right of (above or below) the intersection of these two line segments?", besides the red/blue aboveness testing function, is allowed.

Proof. Let $s(x, y, x_1, y_1)$ be a line segment connecting (x, y) and $(x + x_1, y + y_1)$ in the Euclidean plane. We construct a set of 2D line segments $G_m = \{s'_{ij} | s'_{ij} = s(i + \alpha'_{ij}, j + \beta'_{ij}, \gamma'_{ij}, \delta'_{ij}) | 1 \leq i, j \leq m$ and $\alpha'_{ij}, \beta'_{ij}, \gamma'_{ij}, \delta'_{ij} > 0\} \cup \{s''_{ij} | s''_{ij} = s(i + \alpha''_{ij}, j + \beta''_{ij}, \gamma''_{ij}, \delta''_{ij}) | 1 \leq i, j \leq m$ and $\alpha''_{ij}, \beta''_{ij}, \gamma''_{ij}, \delta''_{ij} > 0\}$. Moreover we can choose $\alpha'_{ij}, \beta'_{ij}, \gamma'_{ij}, \delta'_{ij}$, $\alpha''_{ij}, \beta''_{ij}, \gamma''_{ij}$ and δ''_{ij} so that s'_{ij} intersects s''_{ij}. In other words, at the upper right corner of each point in M there are two tiny intersecting segments. ($\alpha'_{ij}, \beta'_{ij}, \gamma'_{ij}, \delta'_{ij}$, $\alpha''_{ij}, \beta''_{ij}, \gamma''_{ij}$ and δ''_{ij} can all be made very very small.) It is clear that $|G_m| = 2m^2$.

We construct a set of 3D line segments $\mathcal{F}_m$ by first taking an arbitrary set of 3D lines whose vertical projections corresponding the set of lines in F_m and then we translate these segments one by one along the -Z direction so that there is no cyclic overlap among these segments. For each point (i,j) in M we associate a set $S(i,j)$ which contains all the lines in $\mathcal{F}_m$ whose planar projection pass through (i,j). Clearly, $\sum_{i,j}|S(i,j)| \geq m^{8/3}$.

Let $s(x,y,z,x_1,y_1,z_1)$ be a 3D line segment connecting (x,y,z) and $(x+x_1,y+y_1,z+z_1)$ in the Euclidean space. We construct another set of 3D line segments $\mathcal{G}_m$ by mapping each s'_{ij} to $s(i+\alpha'_{ij}, j+\beta'_{ij}, z1'_{ij}, \gamma'_{ij}, \delta'_{ij}, z2'_{ij})$ and each s''_{ij} to $s(i+\alpha''_{ij}, j+\beta''_{ij}, z1''_{ij}, \gamma''_{ij}, \delta''_{ij}, z2''_{ij})$, where $z1'_{ij}, z2'_{ij}, z1''_{ij}, z2''_{ij}$ are arbitrary reals. The set $\mathcal{G}_m$ contains all these mapped segments. Clearly the vertical projections of the line segments in $\mathcal{G}_m$ correspond to those 2D segments in G_m.

The input to the original problems is set as $\mathcal{F}_m \cup \mathcal{G}_m$. Now the adversary controls the segments $\mathcal{G}_m$ by selecting suitable $\alpha'_{ij}, \beta'_{ij}, \gamma'_{ij}, \delta'_{ij}, \alpha''_{ij}, \beta''_{ij}, \gamma''_{ij}$ and δ''_{ij}, for all i,j, so that no segment in $\mathcal{F}_m$ is above or below those in $\mathcal{G}_m$ (i.e., their vertical projections have no intersection). To report this, any algorithm must compute, for all i,j, the aboveness testing functions between $s(i+\alpha'_{ij}, j+\beta'_{ij}, z1'_{ij}, \gamma'_{ij}, \delta'_{ij}, z2'_{ij})$ $(s(i+\alpha''_{ij}, j+\beta''_{ij}, z1''_{ij}, \gamma''_{ij}, \delta''_{ij}, z2''_{ij}))$ and all the line segments in $S(i,j)$. If an algorithm fails to compute any of these functions, the adversary first perturbs $\alpha'_{ij}, \beta'_{ij}, \gamma'_{ij}, \delta'_{ij}$, $\alpha''_{ij}, \beta''_{ij}, \gamma''_{ij}$ and δ''_{ij} so that s'_{ij} and s''_{ij} all intersects with the vertical projection of this line segment l^*_{ij} in $S(i,j)$. Then, the adversary perturbs $z1'_{ij}, z2'_{ij}$, $z1''_{ij}, z2''_{ij}$ so that $s(i+\alpha'_{ij}, j+\beta'_{ij}, z1'_{ij}, \gamma'_{ij}, \delta'_{ij}, z2'_{ij})$, $s(i+\alpha''_{ij}, j+\beta''_{ij}, z1''_{ij}, \gamma''_{ij}, \delta''_{ij}, z2''_{ij})$ and l^*_{ij} forms a cyclic overlap (Figure 1). This holds for all i,j since all the values perturbed are independent with each other for all i,j, and moreover, after this perturbation the algorithm can not distinguish the two input with the aboveness testing function and the primitive "Is this point to the left or right of (above or below) the intersection of these two line segments?" [3]. Consequently, computing the depth order of a set of $O(m^2)$ segments in $\mathcal{F}_m \cup \mathcal{G}_m$ takes at least $\sum_{i,j}|S(i,j)| \geq m^{8/3}$ time.

Finally, to complete the proof of the theorem we simply set $n = m^2$ and make the lines in $\mathcal{F}_m$ into line segments. □

[3] Again, if the primitive "Is the slope of this line larger or smaller than the slope of the line connecting these two points" is allowed then the algorithm can detect the perturbation and therefore this proof fails.

It should be noted that similar to the claim in the last section, this proof can not be immediately generalized to prove the lower bound of computing the depth order of a set of lines in 3D.

6 Concluding Remarks

We mention some related problems as a closing remark. An interesting question is whether one can establish the lower bounds (better than $\Omega(n \log n)$) for these problems under a stronger model, e.g., by allowing more primitives. Although the ultimate objective is to establish these lower bounds under the algebraic decision tree model we feel this is a very hard problem. Another question is whether we can prove other closely related problems under the model used in this paper. With respect to the result of Theorem 3, it should be noted that the constructive proof can not be immediately generalized to establish the lower bounds for the following related problems.

(1) Compute the *longest* red/blue vertical distance between two sets of red and blue line segments each with size $O(n)$.

(2) Compute the shortest red/blue vertical distance between two sets of red and blue *lines* each with size $O(n)$.

(3) Compute the shortest red/blue vertical distance between a pair of red and blue *polyhedral terrains* each with size $O(n)$.

These problems are very closely related to the problem we study in Theorem 3; in fact, the best known upper bounds for solving these problems are either $O(n^{4/3+\epsilon})$ or a little bit higher. Therefore it is interesting to ask whether one can find the lower bound proof for the above problems (and those mentioned in Section 4 and 5: testing the towering property of red/blue *lines* and computing the depth order of 3D *lines*) under the model used in this paper.

Very recently Erickson used the method of [GO93] to attack the lower bounds of some related geometric problems [Eri95b]. He showed that many of the problems are "harder than" or "almost harder than" some others. But there is no single base problem to which all those problems can be reduced. We note that our model is hence different from that of [Eri95b].

References

[CEG+90] B. Chazelle, H. Edelsbrunner, L. Guibas,

R. Pollack, R. Seidel, J. Snoeyink, and M. Sharir. Counting and cutting cycles of lines and rods in space. In *Proc. 31st FOCS*, pages 242–251, 1990.

[CEGS89] B. Chazelle, H. Edelsbrunner, L. Guibas, and M. Sharir. Line in space-combinatorics, algorithms and applications. In *Proc. 21st STOC*, pages 382–393, 1989.

[Cha89] B. Chazelle. Lower bounds on the complexity of polytope range searching. *J. Amer. Math. Soc.*, 2:637–666, 1989.

[Cha90a] B. Chazelle. Lower bounds for orthogonal range searching, I: the reporting case. *J. ACM*, 37:200–212, 1990.

[Cha90b] B. Chazelle. Lower bounds for orthogonal range searching, II: the arithmetic model. *J. ACM*, 37:439–463, 1990.

[Cha94] B. Chazelle. Computational geometry: A retrospective. In *Proc. 26th Annu. ACM Sympos. Theory Comput.*, pages 75–94, 1994.

[CR92] B. Chazelle and B. Rosenberg. Lower bounds on the complexity of simplex range reporting on a pointer machine. In *Proc. 19th International Colloquium on Automata, Languages, and Programming*, volume 623 of *Lecture Notes in Computer Science*, pages 439–449. Springer-Verlag, 1992. Also to appear in *Comput. Geom. Theory Appl.*

[dBOS92] M. de Berg, M. Overmars, and O. Schwarzkopf. Computing and verifying depth orders. In *Proc. 8th Annu. ACM Sympos. Comput. Geom.*, pages 138–145, 1992.

[DS65] H. Davenport and A. Schinzel. A combinatorial problem connected with differential equations. *Amer. J. Math.*, 87:684–689, 1965.

[Eri95a] J. Erickson. New lower bounds for Hopcroft's problem. In *Proc. 11th Annu. ACM Sympos. Comput. Geom.*, pages 127–137, 1995.

[Eri95b] J. Erickson. On the relative complexities of some geometric problems. In *Proc. 7th Canadian Conf. on Comput. Geom.*, pages 127–137, 1995.

[ES93] J. Erickson and R. Seidel. Better lower bounds on detecting affine and spherical degeneracies. In *Proc. 34th Annu. IEEE Sympos. Found. Comput. Sci. (FOCS 93)*, pages 528–536, 1993.

[Fre81] M. L. Fredman. Lower bounds on the complexity of some optimal data structures. *SIAM J. Comput.*, 10:1–10, 1981.

[GO93] A. Gajentaan and M. Overmars. On a class of $O(n^2)$ problems in computational geometry. Report 1993-15, Department of Comput. Sci., Utrecht Univ., 1993.

[GP92] L. Guibas and M. Pellegrini. New algorithmic results for lines-in-3-space problems. Technical Report TR-92-005, International Computer Science Institute, 1992.

[HS86] S. Hart and M. Sharir. Nonlinearity of Davenport-Schinzel sequences and of generalized path compression schemes. *Combinatorica*, 6:151–177, 1986.

[HW65] G. Hardy and E. Wright. *The Theory of Numbers, Fourth Edition.* Oxford University Press, London, 1965.

[Mat93] J. Matoušek. Range searching with efficient hierarchical cuttings. *Discrete Comput. Geom.*, 10(2):157–182, 1993.

[PS85] F.P. Preparata and M.I. Shamos. *Computational Geometry: An Introduction.* Springer-Verlag, 1985.

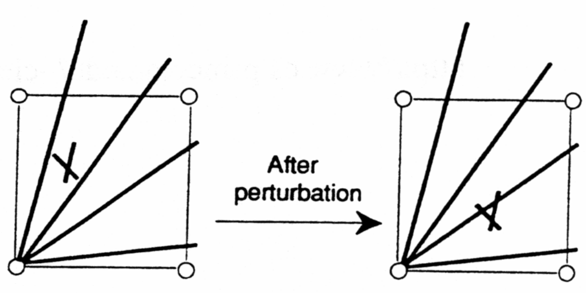

Figure 1. Illustration for the proof of Theorem 5.