On Rectangle Visibility Graphs
III. External Visibility and Complexity

extended abstract

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1 Introduction and Definitions

Let $\mathcal{R} = \{R_i\}$ be a collection of pairwise disjoint closed rectangles in the plane. Two rectangles R_i and R_j will be called *visible* if there is a closed nondegenerate rectangular region B_{ij} (called a *band of visibility*) such that one side of B_{ij} is contained in a side of R_i, the opposite side of B_{ij} is contained in a side of R_j, and B_{ij} does not intersect the interior of any rectangle in $\mathcal{R}$. This type of visibility is equivalent to what Tamassia and Tollis have called ϵ-visibility [8]. The *visibility graph of* $\mathcal{R}$ is the graph of the visibility relation on vertex-set $\mathcal{R}$; a collection of rectangles and its visibility graph is shown in Figure 1. A graph is called a *rectangle visibility graph*, or *RVG*, if it is the visibility graph of some collection $\mathcal{R}$ of rectangles ($\mathcal{R}$ is called the *layout* of the graph). A graph is called a *weak RVG* if it is a subgraph of an RVG. In weak layouts we are allowed to embed two nonadjacent vertices as rectangles that are visible; in non-weak layouts this is not allowed. For convenience, we will henceforth write the prefix $\epsilon-$ to mean "non-weak" when we want to emphasize this aspect (e.g. an $\epsilon-$RVG is a (non-weak) RVG as defined above).

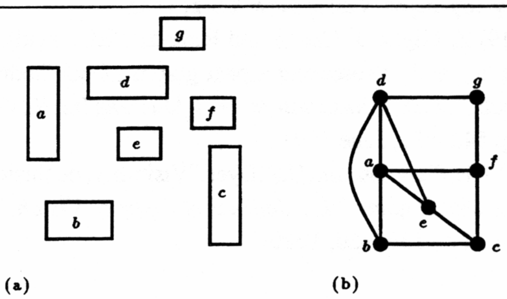

Figure 1: A set of rectangles and its visibility graph

We distinguish between two types of $\epsilon-$layouts of graphs: *collinear* and *noncollinear.* A layout is called collinear if there is a pair of rectangles that have collinear sides, and noncollinear otherwise. This distinction is unnecessary for weak RVGs, as any weak collinear layout can be converted to a weak noncollinear layout by perturbation. Observe that every noncollinear RVG is a collinear RVG, and every collinear RVG is a weak RVG.

Rectangle visibility layouts of graphs are of interest for a variety of reasons. They naturally arise in two-layer routing problems in VLSI or printed-circuit board design. In a more general setting, they can be used for labelled graph layouts, as vertex labels can be written inside the rectangles, and edge labels need only avoid horizontal and vertical obstacles (the sides of the rectangles and the visibility edges). Furthermore, all edge crossings in a rectangle visibility layout are perpendicular, and if the graph is an $\epsilon-$RVG, the crossings in the drawing can eliminated by simply not drawing the edges (the vertex placement defines the graph).

RVGs are closely related to *bar-visibility graphs* (or BVGs): those graphs that can be drawn so that their vertices are represented by horizontal line segments, and their edges by vertical bands of visibility. Any BVG must be planar, as can be seen by shrinking the horizontal segments to points while deforming the rest of the plane, allowing the edges to bend.

We use the modifiers *$\epsilon-$, weak, collinear,* and *noncollinear* on BVGs with the same meaning as for RVGs. Noncollinear BVGs were characterized by Luccio, Mazzone, and Wong [5] as *ipo-triangular* graphs: those graphs that can be transformed into a triangulated planar multigraph by duplications of existing edges. Collinear BVGs were characterized

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by Wismath [9], and independently by Tamassia and Tollis [8], as those planar graphs that can be drawn in the plane with all cut-vertices on a single face. Deciding if a graph is a collinear BVG takes only linear time [8, 10]. Weak BVGs were characterized by Duchet *et al.* as planar graphs [3], and are thus also recognizable in linear time.

RVGs do not seem to have such succinct characterizations. We proceed to survey what is known about RVGs; this review is interspersed with the necessary definitions.

We will say that a graph G can be *decomposed* into graphs G_1 and G_2 if G, G_1, and G_2 have the same vertex set, and the edge set of G is exactly the union of the edge sets of G_1 and G_2. This can be viewed as coloring the edges of G with 2 colors so that one color class forms G_1 and one forms G_2. We will use the concepts of decomposition and edge-coloring interchangably.

By the term *colored graph* we mean a graph whose edges have been colored red and blue. Any RVG can be decomposed into two BVGs by finding a layout of the RVG and coloring edges corresponding to horizontal visibilities red and those corresponding to vertical visibilities blue. If we are given a colored graph, we will say that a rectangle visibility layout of the graph *respects the coloring* if the colors correspond to the directions of visibility as above.

A graph is called *thickness-two* if it can be decomposed into two planar graphs. Any RVG is thickness-two, but not every thickness-two graph is an RVG. In fact, Hutchinson, Shermer, and Vince have shown that any type of RVG has at most $6n - 20$ edges (whereas thickness-two graphs can have as many as $6n - 12$) [4].

Dean and Hutchinson [2] established that the complete bipartite graphs $K_{p,q}$ are noncollinear RVGs iff $p < 3$ or $(p, q) = (3, 3)$ or $(p, q) = (3, 4)$, and that they are collinear iff $p \leq 4$. They also show that a bipartite RVG can have at most $4n - 12$ edges. Wismath has shown that all planar graphs are RVGs [10].

A *caterpillar* is a tree containing a simple path $P(a, b)$ such that every vertex not on $P(a, b)$ is distance one from $P(a, b)$. A *caterpillar forest* is a forest, where each tree of the forest is a caterpillar. Similarly, a *linear forest* is a forest where each tree is a path.

The *arboricity* of a graph G is the minimum k such that G can be decomposed into k forests. Similarly, the *linear arboricty* (or *caterpillar arboricity*) or a graph is the minimum k such that the graph can be decomposed into k linear forests (or caterpillar forests). Note that a graph with linear arboricity k has caterpillar arboricity at most k.

Bose *et al.* have shown that any graph with caterpillar arboricity 2 is a rectangle-visibility graph [1], and ask if such graphs are easily recognizable. Graphs with linear arboricity 2 are included in this result, but their layouts have special properties, and so recognizing this subclass is also of interest. Peroche [6] has proven that recognizing *multigraphs* with linear arboricity 2 is NP-complete, and here we show how to modify Peroche's proof to establish that recognizing *graphs* with linear arboricity 2 is NP-complete, and that recognizing graphs with caterpillar arboricity 2 is NP-complete. This settles the questions raised in [1]. Bose *et al.* also established that the class of noncollinear RVGs includes k-trees, for $k \leq 4$, and partial 2-trees.

Graphs with a maximum vertex degree of 3 have linear arboricity two, so the Bose *et al.* result also shows that these graphs are RVGs. Shermer extended this to show that graphs whose high-degree vertices (degree 4 or more) are far apart are RVGs, and also that maximum-degree 4 graphs are weak RVGs [7].

In this paper, we introduce the notion of external visibility for a BVG or RVG, and characterize some restricted classes of externally visibile BVGs and RVGs. One of these characterizations is essential for establishing our main result, which is that determining if a given graph is an RVG (of any sort—weak, collinear, or noncollinear) is NP-complete. One can view this result as justifying the current approach of trying to find large recognizable subclasses of RVGs rather than trying to find a recognition algorithm or characterization for all RVGs. As noted above, we also show that the problem of recognizing graphs with linear arboricity 2 is NP-complete, as is the problem of recognizing graphs with caterpillar arboricity 2.

2 Externally Visible RVGs

In this section, we study the graphs that are representable when the vertices of a layout are required to be visible from outside the layout in one or more directions; we call such graphs *externally visible.*

In a bar-visibility layout, we will say that a horizontal line segment (i.e. a vertex) w is *N-visible* if a line segment N placed above ("to the north of") the entire layout is visible to w. Similarly, we will say that w is *S-visible* if a line segment S placed below ("to the south of") the entire layout is visible to w. We give similar definitions of N-, E-, S-, and W-visible for RVGs.

Let x be one of N, E, S, or W. If every vertex of a BVG or RVG layout is x-visible, we call the layout a *x-layout,* A BVG or RVG is called a *x-BVG* or *x-RVG*, respectively, if it has an x-layout. We can

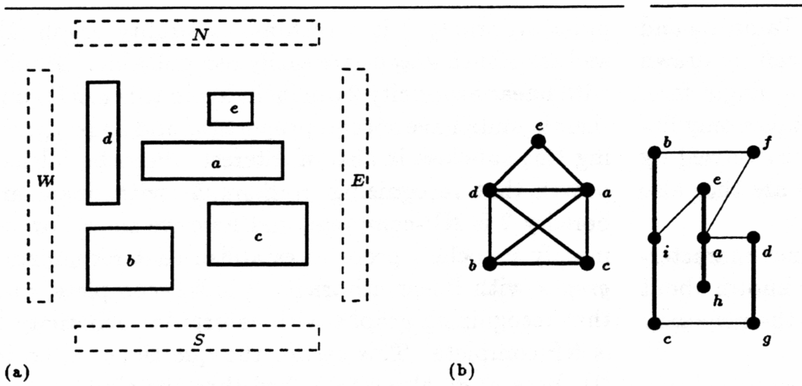

Figure 2: An NE-layout and its visibility graph

repeat the above definitions for x being a subset of {N, E, S, W}, with conjunctive meaning; e.g. *NE-visible* means both N-visible and E-visible. Figure 2 shows an NE-layout and an NE-RVG.

We now characterize N-BVGs and NS-BVGs. The proofs for N-BVGs are omitted.

Theorem 2.1 *A graph G is a weak or collinear N-BVG iff it is outerplanar.*

Theorem 2.2 *A graph G is a noncollinear N-BVG iff each of its two-connected components is a maximal outerplanar graph.*

The following theorem applies to all three types (noncollinear, collinear, and weak) of NS-BVGs.

Theorem 2.3 *A graph G is an NS-BVG iff it is a linear forest.*

PROOF (sketch) If G is a NS-BVG, lay it out and add N and S from the N- and S- visibility definitions. The graph G contains no K_3 because that along with N and S would form a K_5. Furthermore, it cannot contain a cycle because that would imply the existence of a K_3 homeomorph in the layout; thus G is a linear forest. The other direction is by construction; two examples are shown on the left and bottom sides of Figure 3b. □

The preceeding three theorems have ramifications for x-RVGs; e.g. an NES-RVG can be decomposed into a linear forest (an NS-BVG) and an outerplanar graph (a N-BVG). We use this to help establish a characterization for NESW-RVGs; this characterization will be used later in our proof that RVG recognition is NP-complete. We leave exact characterization of the other classes of x-RVGs as open problems.

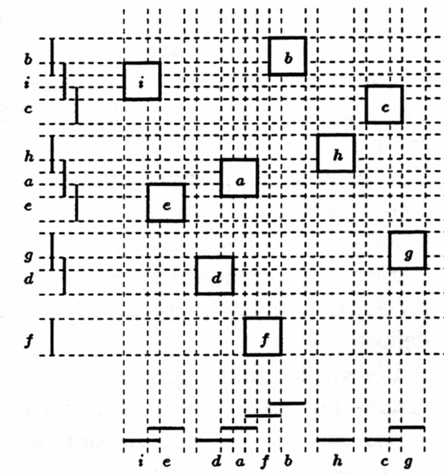

Figure 3: an NSEW-layout of a graph with linear arboricity two

Theorem 2.4 *A graph G is an NESW-RVG iff it has linear arboricity 2.*

PROOF If G is an NESW-RVG, then it has linear arboricity 2, by Theorem 2.3. On the other hand, if G has linear arboricity 2, then the algorithm of Bose *et al.* [1] will produce NESW-layout of G, if we require it to treat each path in the decomposition of G as a caterpillar with no feet. To obtain this layout, the algorithm constructs a horizontal NS-layout for one linear forest (as in our proof of Theorem 2.3), and a vertical NS-layout for the other; each vertex then is laid out as a rectangle that is the cartesian product of its corresponding intervals in the horizontal and vertical layouts. Figure 3 shows an example of a graph G with linear arboricity two and an NSEW-layout for G. □

We can also consider requiring that each vertex be {NS}-visible: *either* N-visible *or* S-visible. We can establish the following, but suspect that one cannot obtain similar results for RVGs.

Theorem 2.5 *A graph G is a (weak, collinear, or noncollinear) {NS}-BVG iff it is a (weak, collinear, or noncollinear, respectively) N-BVG.*

3 Linear and Caterpillar Arboricity

Let *Linear Arboricity 2* be the problem of determining, for a given graph L, if the linear arboricity of L

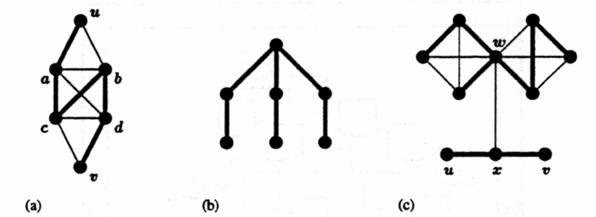

Figure 4: The doubled-edge replacement, forbidden graph, and path component

is 2, and *Linear Arboricity 2 for Multigraphs* be the same problem for a given multigraph. By Theorem 2.4, *Linear Arboricity 2* is equivalent to *NESW-RVG Recognition*, the problem of determining if a given graph is an NESW-RVG. Peroche has shown that *Linear Arboricity 2 for Multigraphs* is NP-complete [6], by transformation from 3-SAT.

We transform an instance M of *Linear Arboricity 2 for Multigraphs* to an instance of *Linear Arboricity 2* by replacing each doubled edge uv with the component shown in Figure 4a. Note that any other multi-edge (e.g. a tripled edge) in M implies the existence of a monochromatic cycle and thus a "no" answer. We can argue that this component acts exactly like a doubled edge in that any decomposition into two linear forests, it must consist of a red path and a blue path between u and v. We therefore obtain:

Theorem 3.1 *Linear Arboricity 2 is NP-complete.*

We are also interested in the problem of determining if a given graph has caterpillar arboricity 2, which we denote *Caterpillar Arboricity 2*. This problem is trivially in NP, and we will show that it is NP-complete by transformation from *Linear Arboricity 2*.

We note that a path is a caterpillar with no branching; if we somehow disallow branching in our caterpillar arboricity coloring of a graph, then we are actually performing a linear arboricity coloring of the graph.

We will disallow branching by replacing each edge uv in the *Linear Arboricity 2* instance L with the component in Figure 4c. This component ensures that the edge wx is a "leg" edge of the caterpillar that it is in, with the path edges of that caterpillar contained in the K_4's containing w. Thus, the edges ux and vx must both have the opposite color; we have essentially subdivided each edge of L and required that the two resulting edges have the same color. This means that if we have any branch point v in the coloring of L (i.e. vertex v with at least three incident edges of the same color), then there is a monochromatic graph as in Figure 4b. However, this graph is not a subgraph of any caterpillar, so branching is disallowed; the caterpillar and linear arboricities must be the same. This establishes the following:

Theorem 3.2 *Caterpillar Arboricity 2 is NP-complete.*

4 Rectangle Visibility Graph Recognition

Let *RVG Recognition* be the problem of deciding if a given graph G is a rectangle visibility graph. In this section, we show that *RVG Recognition* is NP-complete. The proof is by transformation from *Linear Arboricity 2*, and is fairly laborious.

In the first subsection, we study a geometric configuration of four rectangles that we call a *four-way*, and show how to augment a graph so that in any layout, a given four element subset of vertices of the graph form a four-way. In the second subsection, we further constrain the situation to what we call *cyclic four-ways*, and show that under certain conditions, if a graph with a cyclic four-way has a layout, then the subgraph that is "inside" the cyclic four-way must have linear arboricity two. In the third subsection, we present an RVG with a maximal number of edges, and show how to augment it so that in any layout of the augmented graph, the favorable conditions exist for some cyclic four-way. In the final subsection, we present the problem transformation and complete the proof.

Four-ways. Four rectangles are said to form a *four-way* if there is some nondegenerate rectangular region T of the plane with one of the four rectangles to each of its four sides; T is called the *four-way rectangle* for the four rectangles. A four-way is called *visible* if the four rectangles can see a common subset of T.

Suppose we have a subgraph G' with a layout, and a visible four-way S. We will say *place G' in S* to mean that we take some rectangle T' contained in the *interior* of the rectangular region that sees the four elements of the four-way, and scale the layout of G' so as to fit in T' and then to place it there. Since T' is properly contained in this four-way, this process does not destroy any bands of visiblity, and so it does not destroy any visible four-ways.

A *four-way enforcer* for a 4-element subset $S = \{a, b, c, d\}$ of the vertices of a graph is a collection of seventeen K_4's, where each of the 68 vertices in these K_4's is connected to a, b, c, and d. A four-way enforcer has linear arboricity two, and thus has a NESW-layout.

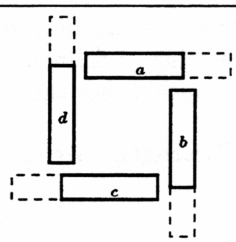

Figure 5: The layout of a cyclic four-way

Lemma 4.1 *If G is a graph containing the vertices $a, b, c,$ and d and a four-way enforcer for $S = \{a, b, c, d\}$, and there is a layout L for G, then the rectangles a, b, c, and d form a four-way.*

Cyclic Four-ways and Included Graphs. We will call a four-way $\{a, b, c, d\}$ in a colored graph *cyclic* if it induces a K_4, and the vertices can be relabeled so that the edges ab, ac, and cd colored blue, and ad, db, and bc colored red. The layout of cyclic four-ways is quite constrained.

Lemma 4.2 *If $\{a, b, c, d\}$ is a cyclic four-way in a graph G, and G is laid out in a way that respects its coloring, then the rectangles $\{a, b, c, d\}$ must be laid out as shown in Figure 5a.*

Given a cyclic four-way $\{a, b, c, d\}$, we obtain a rectangle that we call the *main area* of the four-way by starting with the four-way rectangle and expanding it until it encounters a, b, c, and d. A rectangle that contains no points of the main area will be called an *external rectangle.* An external rectangle will be called *adjacent* if it is visible to a, b, c, and d.

Lemma 4.3 *Let $\{a, b, c, d\}$ be a cyclic four-way in a colored graph M, and let G be the subgraph of M induced by those vertices that are adjacent to all four elements of $\{a, b, c, d\}$ (excluding $a, b, c,$ and d). If M has a rectangle visibility layout that respects the coloring, and such that are no adjacent external rectangles for $\{a, b, c, d\}$ in this layout, then G has linear arboricity 2.*

The main point of the proof of this lemma is that M must have a NESW-layout, and we then apply Theorem 2.4.

The Maximal Graph $M(q)$. We use the construction of Hutchinson, Shermer, and Vince for an RVG $M(q)$ on $n = q^2 + 4$ vertices with the maximum number $6n - 20$ of edges [4]. This graph is defined as the RVG of the arrangement of 4 rectangles e_1, e_2, e_3, and e_4 around q^2 rectangles $v_{i,j}$ placed as shown in Figure 6 for $q = 5$. The figure shows what we call the *standard layout* for this graph; none of the collinearities in this layout are essential and they can be removed by perturbation.

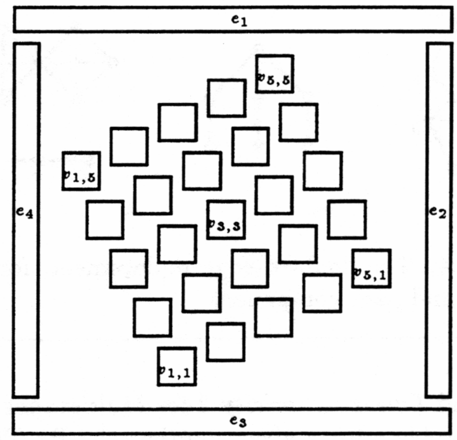

Figure 6: The standard layout of $M(5)$

Let $M'(q)$ be the graph derived from $M(q)$ by adding a four-way enforcer to $M(q)$ for every visible four-way in the standard layout of $M(q)$.

If we have any layout of $M(q)$ (and the corresponding coloring), then we will call a vertex v *special* if (I.) v is $e_1, e_2, e_3,$ or e_4, or adjacent to any of these vertices; (II.) v is on a nontriangular face of either BVG; or (III.) v is on the exterior face of either BVG. A vertex that is not special is called *ordinary*, and an ordinary vertex w is called k-ordinary if there is no special vertex within distance k of w in $M(q)$.

We can establish that there are at most 42 special vertices in any layout of $M(q)$. This implies that for any given k, if q is made large enough, then there will be a k-ordinary vertex in $M(q)$; in particular, if $q \geq 54$, there is a 2-ordinary vertex.

Let $S_{i,j}$ be the four-way $\{v_{i,j}, v_{i,j+1}, v_{i+1,j}, v_{i+1,j+1}\}$. We can proceed to show that for any 1-ordinary vertex v, in any layout of $M'(q)$, the neighborhood of v in $M(q)$ must be colored exactly as it was in the standard layout, and show the following lemma:

Lemma 4.4 *Let M'' be an supergraph of $M'(q)$ and $v = v_{i,j}$ be a 2-ordinary vertex of $M'(q)$ in a layout of M'' (with the corresponding coloring). Then $S_{i,j}$ is a cyclic four-way with no adjacent external rectangles.*

The Transformation. We are now ready to describe how to transform an instance L of *Linear Arboricity 2* to an instance of *RVG Recognition.*

We start by constructing the maximal graph $M = M(54)$, and $M' = M'(54)$, as described in the last section. Complete the construction by including a copy of L adjacent to all vertices of $S_{i,j}$ for each $1 \leq i, j \leq 49$, giving a graph M''.

This transformation is polynomial: M has constant size, M' also has constant size, and M'' is larger than M' only by a constant times the size of L.

Theorem 4.5 *RVG Recognition is NP-complete.*

PROOF *RVG Recognition* is in NP, as one can guess a layout and verify it.

If L has linear arboricity two, then construct a layout of M'' as follows. Lay out M using its standard layout. Next, place each four-way enforcer (using its NESW-layout) in the visible four-way that it enforces. Finally, place each copy of L (using its NESW-layout) in the four-way for its four adjacent vertices. This gives a noncollinear layout for M''.

If M'' has a layout of any variety, then it has a weak layout. There is some vertex $v_{i,j}$ in M (and thus in M'') that is 2-ordinary. By Lemma 4.4, the four-way $S_{i,j}$ is cyclic and has no adjacent external rectangles. But an instance of L is included in M' adjacent to $S_{i,j}$; by Lemma 4.3, L must have linear arboricity two. □

5 Conclusion

In summary, we have introduced several classes of externally visible BVGs and RVGs, and have given characterizations of some of them. Of particular importance (to the current work, at least) is the characterization of NESW-RVGs as graphs with linear arboricity at most 2. We have shown that the problems *Linear Arboricity 2* and *Caterpillar Arboricity 2*, which are closely connected with RVGs, are NP-complete. We have also established that *RVG Recognition* is NP-complete, for weak, collinear, or noncollinear RVGs.

It is still open as to whether or not RVGs have a succinct graph-theoretical characterization; the NP-completeness result here is somewhat discouraging with regards to this. However, *NESW-RVG Recognition* being NP-complete does not stop one from establishing a satisfying characterization of NESW-RVGs (Theorem 2.4), so there is still some hope that RVGs may be characterizable. The other problems that we have left open here are those concerning the different classes of externally visible RVGs that were presented in Section 2.

In closing, the author would like to thank James Abello, Alice Dean, and Joan Hutchinson for many helpful discussions and suggestions.

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