On the Permutations Generated by Rotational Sweeps of Planar Point Sets

Hanspeter Bieri and Peter-Michael Schmidt

Extended Abstract

1 Introduction

The *sweep technique* has proved to be one of the most powerful paradigms in Computational and Combinatorial Geometry, especially when dealing with problems in the 2-dimensional Euclidean plane. In most cases the plane is swept by a straight line whose normal vector never changes its direction. Such a *plane sweep* is called *translational*. In some other cases it is more appropriate to perform a *rotational* plane sweep, i.e. the plane is swept by rotating a straight line or halfline (ray) around a point.

Starting from previous works by J.E. Goodman and R. Pollack ([GoPo80], [GoPo93]) one of the authors studied the following problem ([Schm92]): Let $P = \{p_1, \cdots, p_n\}$ a set of points in the plane $\mathbb{R}^2$ and $G(\tau) \subset \mathbb{R}^2$ a straight line to be used as a *sweep line*. We perform a translational sweep through $\mathbb{R}^2$ by varying the parameter τ from $-\infty$ to $+\infty$ and assume *general position*, i.e. at no moment (i.e. for no value of τ) $G(\tau)$ contains more than one point of P. By doing this we impose an *order* on P and thus generate a *permutation* of the corresponding index set $\{1, \cdots, n\}$ which we denote by π: Let τ_i be the value of τ uniquely determined by $p_i \in G(\tau)$ $(i = 1, \cdots, n)$. We define π by $\pi(i) < \pi(j)$ iff $\tau_i < \tau_j$. [GoPo80] and [Schm92] examine which of the $n!$ orders of P or permutations of $\{1, \cdots, n\}$, respectively, can be generated in this way by translational sweeps. It is shown that their number is in $\Theta(n^2)$, and in [Schm92] an algorithm to find all these orders is given.

In the present paper we study an analogous but more general problem by considering rotational instead of translational sweeps. One common kind of rotational sweeps sweeps the plane by rotating a *straight line* around a fixed point by an angle of 180°, a second kind rotates a *straight halfline* (ray) around its endpoint by an angle of 360°. We will call these two kinds of rotational sweeps R_L*–sweep* and R_H*–sweep*, respectively. They impose different orders on a given set P, in general.

First we give precise definitions of R_L–sweeps and R_H–sweeps and of the quantities we want to determine. We compare the two kinds of sweeps with regard to the permutations they generate for the same set P. Lower and upper bounds on the number of permutations are determined and it is shown that for R_L-sweeps as well as for R_H-sweeps the maximal number of permutations that can be generated in case of n points is in $\Theta(n^4)$. Finally we present for both kinds of rotational sweeps an optimal algorithm which finds for a given set of points all permutations that can be generated.

2 Fundamentals

In order to define more precisely the two kinds of rotational plane sweeps introduced above, we assume $\mathbb{R}^2$ to be equipped with its natural basis and choose any point $q = (x, y) \in \mathbb{R}^2$ as a *center of rotation*. A *R_L-sweep around* q is defined by an angle $\tau_1 \in [0, 2\pi)$ and a *sweep line* $L(q, \tau) = \{(x + r\cos\tau,\ y + r\sin\tau) : r \in \mathbb{R}\}$ depending on a parameter τ which is assumed to run through the interval $[\tau_1, \tau_1 + \pi)$. This means that a straight line whose initial position is defined by the angle τ_1 turns around the point q by an angle of 180°. Every point $p \in \mathbb{R}^2$, $p \neq q$, is swept exactly once.
In an analogous way, a *R_H-sweep around* q is defined by an angle $\tau_1 \in [0, 2\pi)$ and a *sweep halfline* $H(q, \tau) = \{(x + r\cos\tau,\ y + r\sin\tau) : r \in \mathbb{R}_0^+\}$ depending on τ. Here the parameter τ is assumed to run through the interval $[\tau_1,\ \tau_1 + 2\pi)$. This means that a straight halfline whose initial position is defined by τ_1 turns around its endpoint q by an angle of 360°. Again every point $p \in \mathbb{R}^2$, $p \neq q$, is swept exactly once.
Now we consider a nonempty finite set $P \subset \mathbb{R}^2$ being in *general position* in the following sense: Let $\mathbf{H}(P)$ be the set of all straight lines joining two points of P. We assume that no two lines of $\mathbf{H}(P)$ are parallel and no three lines of $\mathbf{H}(P)$ meet in one point. (Every point set P we will consider in this paper is assumed to be in $\mathbb{R}^2$, finite and in general position.) $\mathbf{H}(P)$ partitions the plane $\mathbb{R}^2$ into finitely many relatively open convex *cells* of dimensions $0, 1$ or 2 which represent an *arrangement* $\mathbf{A}(P)$. Let D denote the union of all 2-dimensional cells of $\mathbf{A}(P)$.
By choosing $q \in D$ and performing a R_L-sweep around q we obviously define a *linear order* $<_{L,q}$ on P and a *permutation* $\pi_{L,q}$ of the corresponding index set $\{1, \cdots, n\}$. More precisely, we define them as follows: We denote by τ_i the value of τ uniquely determined by $p_i \in L(q, \tau)$ $(i = 1, \cdots, n)$, which implies – without loss of generality – that the sweep is assumed to start at point p_1. The linear order $<_{L,q}$ imposed on P is then defined by $p_i <_{L,q} p_j$ iff $\tau_i < \tau_j$, and the corresponding permutation $\pi_{L,q}$ of $\{1, \cdots, n\}$ is analogously defined by $\pi_{L,q}(i) < \pi_{L,q}(j)$ iff $\tau_i < \tau_j$. In case of a R_H-sweep, the linear order $<_{H,q}$ imposed on P and the corresponding permutation $\pi_{H,q}$ of $\{1, \cdots, n\}$ are defined analogously. Of course, we could also work with cyclic instead of linear orders.

Some of the following considerations apply to both, R_L-sweeps and R_H-sweeps. In order to indicate that we do not have to examine the two cases separately we will use the symbol S instead of R_L or R_H. By $\Pi_S(P)$ we denote the set of all permutations of $\{1, \cdots, n\}$ which can be obtained by means of S–sweeps applied to P. $\Pi_S(P)$ is assumed to be (arbitrarily) ordered, and permutations in $\Pi_S(P)$ will then just be denoted by π_j or in the form $(\pi_j(1) \cdots \pi_j(n))$ $(j = 1, \cdots, \text{card}(\Pi_S(P)))$ Our assumption that every S–sweep starts at p_1 implies that we actually only look for permutations of $\{2, \cdots, n\}$, i.e. we examine which of $(n-1)!$ possible permutations can be generated by S–sweeps. $\Pi_S(P)$ and $\text{card}(\Pi_S(P))$ do not only depend on the number of points in P, in general, but also on how these points are arranged. Therefore

$$N_S(n) := \max\{\text{card}\,(\Pi_S(P)) : \ \text{card}(P) = n\}$$

is another quantity we are interested in $(n \in \mathbb{N})$.

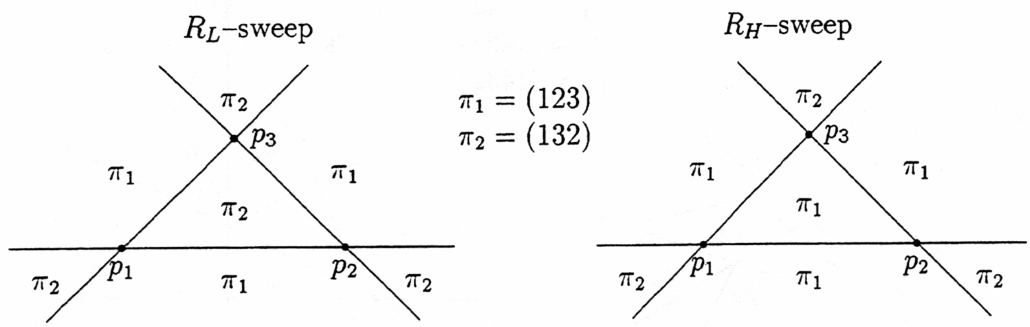

Figure 1: An arrangement $A\{p_1, p_2, p_3\}$ with each of its 2-dimensional cells marked with the corresponding permutation π_j of $\{1, 2, 3\}$, for $S = R_L$ and $S = R_H$.

Given P, it is natural to ask for all points $q \in D$ whose corresponding S–sweeps impose the same order on P. These points form obviously an *equivalence class* of the *equivalence relation* $\sim_S$ defined on D by

$$q \sim_S q' \iff \pi_{S,q} = \pi_{S,q'} \text{ for all } q, q' \in D.$$

It is easily seen that all q belonging to the same 2–dimensional cell are equivalent with respect to $\sim_S$. Hence every equivalence class of $\sim_S$ is the union of certain 2–dimensional cells of $\mathbf{A}(P)$. Sometimes we will indicate an equivalence class of $\sim_S$ just by its corresponding permutation. Figure 1 shows the equivalence classes of $\sim_L$ and $\sim_H$ for a set $P = \{p_1, p_2, p_3\}$ whose corresponding triangle $\Delta(p_1, p_2, p_3)$ is positively oriented. $\Pi_L(P) = \Pi_H(P)$ and $\text{card}(\Pi_S(P)) = 2$, but $\sim_L \neq \sim_H$.

3 Two general results

As already mentioned, R_L–sweeps and R_H–sweeps do not generate the same permutations of $\{1, \cdots, n\}$, in general, i.e. $\Pi_L(P) \neq \Pi_H(P)$. A more precise comparison is given by the following

Theorem 1 *For every P the following assertions hold:*

a) $\Pi_L(P) = \Pi_H(P)$ *if* $card(P) \leq 4$ *(cf. Figure 2).*

b) $\Pi_L(P) \supset \Pi_H(P)$ *if* $card(P) = 5$.

c) Either $\Pi_L(P) \supset \Pi_H(P)$ *or* $(\Pi_L(P) \not\supseteq \Pi_H(P) \wedge \Pi_L(P) \not\subseteq \Pi_H(P))$ *if* $card(P) = n \geq 6$. *For every such n both cases occur.*

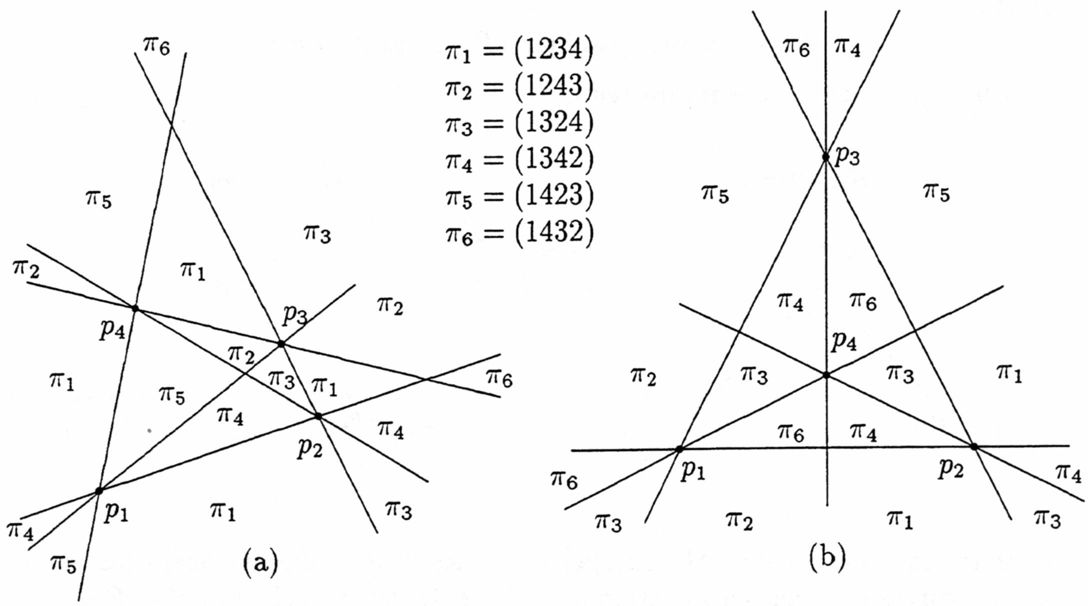

Figure 2: The two possible kinds of arrangements belonging to 4–point sets together with the corresponding permutations generated by R_L–sweeps.

Let $S \in \{R_L, R_H\}$. As already indicated in the introduction, the number of permutations that can be generated by S–sweeps of a set of n points is much smaller than $(n-1)!$, in general. A more precise statement is the following

Theorem 2 $N_S(n) \in \Theta(n^4)$ *for every* $n \in I\!N$.

4 An algorithm for finding $\Pi_S(P)$

Let $n > 1$ and given $P = \{p_1, \cdots, p_n\} \subset I\!R^2$ in general position. $\mathbf{H}(P)$ consists of $\binom{n}{2} \in \Theta(n^2)$ lines. Every 2–dimensional cell $C \in \mathbf{A}(P)$ is the intersection of open halfplanes belonging to lines of $\mathbf{H}(P)$. C can be specified, therefore, e.g. by providing for each such halfplane a number identifying the corresponding line of $\mathbf{H}(P)$ and a sign which indicates if the halfplane is positive or negative. We conceive all these 2–dimensional cells of $\mathbf{A}(P)$ as nodes of a graph $G(P)$. Two nodes of $G(P)$

are joined by an edge iff there exists a 1–dimensional cell in $\mathbf{A}(P)$ which is adherent to both 2–dimensional cells represented by them.
Figure 3 shows $\mathbf{A}(P)$ and $G(P)$ for a set P of only three points. The graph $G(P)$ contains $\Theta(n^4)$ nodes and is closely related to the well–known *incidence graph* $G^*(P)$ introduced by [EdOS86] to represent $\mathbf{A}(P)$. As $G^*(P)$ can be constructed in optimal time $\Theta(n^4)$ (cf. [EdOS86]) the same is true for $G(P)$.
The graph $G(P)$ can be constructed incrementally, using the same ideas as the construction of $G^*(P)$.
Next we perform a *depth–first traversal* of $G(P)$ which costs $\Theta(n^4)$ time. That is, we traverse all 2–dimensional cells $C \in \mathbf{A}(P)$, hence find all equivalence classes of $\sim_S$. The permutation belonging to the first cell we traverse can be found in $\Theta(n \cdot \log n)$ time by determining a point $q \in C$ and executing the corresponding S–sweep. The permutations of any two incident cells differ only by a transposition. Hence, by reporting just these transpositions we can find the permutation of any cell (except the first one) from the permutation of the previous one in constant time. In this way, we can find all permutations – in general more than once – in $\Theta(n^4)$ time.
Considering that $\text{card}(\Pi_S(P)) \in \Theta(n^4)$ (Theorem 2) we conclude:

Theorem 3 *For a n–point set P, $\Pi_S(P)$ can be found in time $\Theta(n^4)$, which is optimal.*

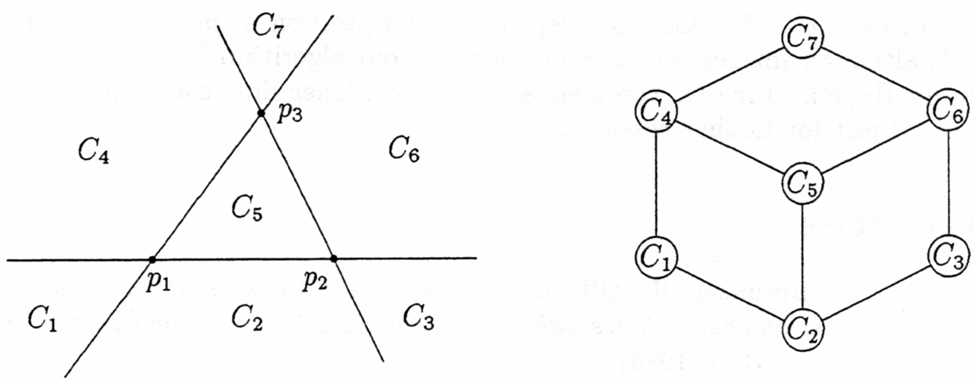

Figure 3: The arrangement $\mathbf{A}(P)$ and the graph $G(P)$ for $P = \{p_1, p_2, p_3\}$.

We summarize our approach in form of an algorithm FINDPERM:

input: $P = \{p_1, \cdots, p_n\} \subset \mathbb{R}^2$ in general position. $S \in \{R_L, R_H\}$.
output: All permutations generated by S–sweeps.

procedure FINDPERM

1. Construct the graph $G(P)$.

2. For any 2–dimensional cell $C \in \mathbf{A}(P)$ choose $q \in C$ and find $\pi_{S,q}$.

3. Depth–first traversal of $G(P)$, starting at C.
   For each traversed edge, find the corresponding transposition.

4. Optionally: Report complete permutations.

5. Optionally: Eliminate duplicates.

6. Optionally: Report for each $\pi_j \in \Pi_S(P)$ the list $C(\pi_j)$ of all 2–dimensional cells $C \in \mathbf{A}(P)$ belonging to π_j.

end FINDPERM

5 Outlook

From several points of view it seems to be a challenging problem to characterize geometrically the equivalence classes of $\sim_L$ and $\sim_H$, respectively, probably in the context of projective geometry. It also would certainly be interesting to characterize (for $n \geq 6$) those permutations π which are in $\Pi_L(P) \cap \Pi_H(P)$, and those P for which $\Pi_L(P) \supset \Pi_H(P)$ holds. It might be worthwhile to examine also

$$M_S(n) := \min\{\mathrm{card}(\Pi_S(P)) : \mathrm{card}(P) = n\}.$$

Our conjecture is, that $M_S(n) \in \Theta(n^4)$, which would mean that the order of magnitude of card($\Pi_S(P)$) does not depend on the position of $p_1, \cdots, p_n$. But such a result, although interesting, should not affect our algorithm FINDPERM for determining $\Pi_S(P)$. Finally, the generalization to higher dimensions represents an obvious subject for further research.

References

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