Minimal Tangent Visibility Graphs

Extented Abstract

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Abstract

We prove the tight lower bound $4n-4$ on the size of tangent visibility graphs on n pairwise disjoint bounded obstacles in the Euclidean plane. We give also a simple characterization of the set of minimal tangent visibility graphs.

1 Introduction

Visibility and shortest path problems in a scene consisting of disjoint polygons in the plane have been studied extensively. Recently the scope of this research has been extended to scenes of disjoint convex plane sets (convex obstacles for short). One of the combinatorial questions concerns the complexity of such scenes. Our starting point is the following problem: what is the minimal number of *free bitangents* shared by n convex obstacles? A *bitangent* is a closed line segment whose supporting line is tangent to two obstacles at its endpoints; it is called *free* if it lies in *free space* (i.e., the complement of the union of the relative interiors of the obstacles). The endpoints of these bitangents split the boundaries of the obstacles into a sequence of arcs; these arcs and the bitangents are the edges of the so-called tangent visibility graph. The size of the tangent visibility graph is defined to be the number of free bitangents, so our question asks for the minimal size of tangent visibility graphs. Visibility graphs (for polygonal obstacles) were introduced by Lozano-Perez and Wesley [8] for planning collision-free paths among polyhedral obstacles; in the plane a shortest Euclidean path between two points runs via edges of the tangent visibility graph of the collection of obstacles augmented with the source and target points. Since then numerous papers have been devoted to the problem of their efficient construction ([19, 6, 1, 4, 7, 10, 3, 18, 14, 15]) as well as their characterization (see [9] and the references cited therein). The more recent papers [11, 13, 12] consider the problem of the efficient computation of tangent visibility graphs for curved obstacles. This paper is concerned with the problem of characterizing these graphs in the case where they have minimal size. The answer to our question is given in the following theorem (we assume that the obstacles are not reduced to points).

Theorem 1 *The number of free bitangents shared by n pairwise disjoint convex obstacles is at least* $4n-4$*; this bound is tight.* □

Configurations of $n(=4)$ convex obstacles with exactly $4n-4(=12)$ bitangents are depicted in Figure 1. These examples are easily extented for any value of n. The $4n-4$ lower bound has been established previously in the case where the obstacles are line segments by [16] (see also [2, 17]). We give here a different proof based on the notion of *pseudo-triangulation* introduced in [11]. In fact we prove the following stronger result.

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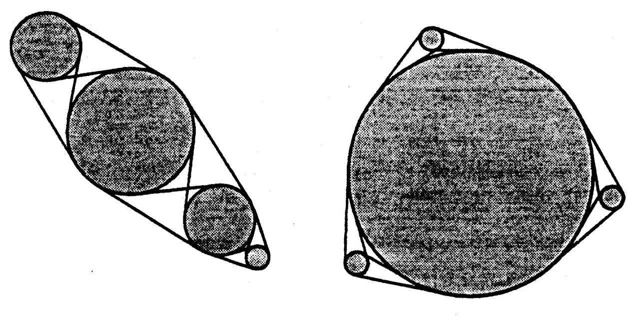

Figure 1: Configurations of 4 obstacles with $4 \times 4 - 4 = 12$ free bitangents.

Theorem 2 *Consider a collection of n pairwise disjoint convex obstacles. The following assertions are equivalent.*

1. *Its weak visibility graph is a tree.*
2. *Its number of free bitangents is minimal (i.e., $4n-4$).*
3. *The size of its convex hull is maximal (i.e., $2n-2$).* □

Recall that the *weak visibility graph* is the graph whose nodes are the obstacles and whose edges are pairs of obstacles which are *weakly visible*, i.e., there is a free line segment whose endpoints belong to the obstacles. The *size* of the convex hull is the number of bitangents appearing on its boundary.

To give a characterization of minimal tangent visibility graphs we use the notion (introduced in [12]) of *visibility type* of a configuration of obstacles; it may be considered as a combinatorial version of the tangent visibility graph where we take into account the circular order of the free bitangents around each obstacle. More precisely, let $O = \{O_1, \ldots, O_n\}$ be a collection of n pairwise disjoint, not necessarily convex, obstacles. We define the *canonical label* of a bitangent directed from obstacle O_i to obstacle O_j to be the symbol $(\epsilon i, \epsilon' j)$ with $\epsilon = +$ or $-$ ($\epsilon' = +$ or $-$) depending on whether O_i (O_j) lies, locally at the touch point, to the left or to the right of the directed bitangent. Consider now the set of free bitangents to O_i directed in such a way that O_i lies (locally) to the left of the bitangent. These free directed bitangents can be ordered counterclockwise around the obstacle O_i. The circular sequence of canonical labels of these bitangents is called the (visible) cycle $C(i)$. Note, e.g., that the cycle $C(i)$ of a collection of two convex obstacles O_i and O_j is $C(i) = (i,j)(i,-j)(-j,i)(j,i)$. The collection of n cycles $C(1), C(2), \ldots, C(n)$ is called the visibility type of the collection O and is denoted by $C(O)$.

Theorem 3 *The set of minimal visibility types on n disjoint convex obstacles is in 1-1 correspondence with the set of plane labeled trees on n nodes.* □

For the sake of simplicity we assume that each obstacle is *strictly* convex and has a *smooth* boundary (however our results are still valid without these assumptions). An *extremal* point of an obstacle is a boundary point at which the tangent line to the boundary is horizontal.

The paper is organized as follows. In section 2 we introduce the notion of pseudo–triangulation and we prove three technical lemmas. In section 3 we prove our main results. Finally in section 4 we generalize our results to configurations of disjoint non-convex obstacles; we prove that $4n-4$ is still a tight lower bound for the number of free bitangents of a collection of n disjoint obstacles, and we give a simple caracterization of the corresponding minimal visibility types. Due to the lack of space we omit most of the proofs in this version of the paper.

2 Pseudo–triangulation

A *pseudotriangle* is a simply connected bounded subset R of $\mathcal{R}^2$ such that (i) the boundary ∂R is a sequence of three convex curves that are tangent at their endpoints, and (ii) R is contained in the triangle formed by the three endpoints of these convex curves (see Figure 2). A *pseudo–triangulation* of the set of obstacles is the subdivision of the plane induced by a maximal (with respect to the inclusion relation) family of pairwise noncrossing free bitangents. It is clear that a pseudo–triangulation always exists and that

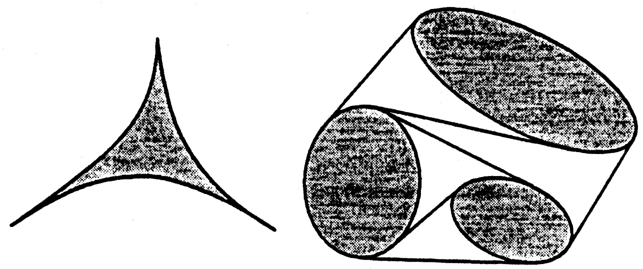

Figure 2: A pseudotriangle and a pseudo–triangulation.

the bitangents of the boundary of the convex hull of the obstacles are edges of any pseudo–triangulation. A pseudo–triangulation of a collection of three obstacles is depicted in Figure 2.

Lemma 1 *The bounded free faces of any pseudo–triangulation are pseudotriangles.* □

Lemma 2 *Consider a pseudo–triangulation of a collection of n disjoint convex obstacles induced by a maximal family B of free bitangents and let F_i be the set of pseudotriangles with exactly i bitangents on their boundaries. Then we have*

$$
\begin{aligned}
|B| &= 3n-3 && (1)\\
|F_2|+|F_3|+\cdots &= 2n-2 && (2)\\
2|F_2|+3|F_3|+\cdots &= 6n-6-h && (3)\\
|F_3|+2|F_4|+\cdots &= 2n-2-h && (4)
\end{aligned}
$$

where h is the number of bitangents on the boundary of the convex hull of the collection. □

From equation (4) we deduce that $2n-2$ is an upper bound for h; Figure 1 shows that this upper bound is tight. An alternative argument is the following. The number h is also the size of the circular sequence of obstacles that appear on the convex hull (we call this sequence the *combinatorial* convex hull of the collection of obstacles). Since the obstacles are pairwise disjoint this circular sequence is a circular Davenport-Schinzel sequence on n symbols and parameter 2, (i.e., factors *aa* and subwords *abab* are forbidden). It is well-known (and easy to verify) that such a circular sequence has length at most $2n-2$. Conversely any circular Davenport-Schinzel sequence (not necessarily maximal) on n symbols with parameter 2 can be realized as the combinatorial convex hull of n pairwise disjoint obstacles. The argument is very simple. Let $i_1 \ldots i_h$ be a circular Davenport-Schinzel sequence on the alphabet $\{1, \ldots, n\}$ with parameter 2. Now label in clockwise order the h vertices of a regular h-gon by the indices of the sequence $i_1 \ldots i_h$. The convex hulls O_i of the points labeled i are pairwise disjoint (because subwords *abab* are forbidden) obstacles whose combinatorial convex hull is exactly $i_1 \ldots i_h$. Finally we note the following simple fact.

Lemma 3 *Consider a pseudo–triangulation of a collection of obstacles, and let F_2 be the set of pseudotriangles with exactly 2 bitangents on their boundaries. Then a pseudotriangle in F_2 is adjacent to at most one other pseudotriangle in F_2.* □

3 Proof of the main results

We begin with a lemma.

Lemma 4 *The number of free bitangents of a collection of n disjoint convex obstacles is at least $6n-6-h$, where h is the number of bitangents on the boundary of the convex hull of the collection.* □

Proof of Theorem 2. Since the number of bitangents between two convex obstacles is 4 it is clear that the size of a tangent visibility graph is bounded above by 4 times the number of edges of the weak visibility graph. Assuming (1) (i.e., the weak visibility graph is a tree) it follows that the size of the tangent visibility graph is bounded above by $4n-4$; since $4n-4$ is a lower bound the size of the tangent visibility graph is exactly $4n-4$. This proves that (1) implies (2). Now (2) $\Rightarrow$ (3) is an obvious consequence of lemma 4 and the fact that $2n-2$ is an upper bound for the size of the convex hull. Now we prove that (3) implies (1). According to equation (4) of Lemma 2 we have $|F_i| = 0$ for $i \geq 3$, i.e. the $2n-2$ pseudotriangles of any pseudo–triangulation have exactly two bitangents on their boundaries. It follows (see Lemma 3) that the connected components of bounded free space are pseudoquadrangles (i.e., the union of two adjacent pseudotriangles). There are $n-1$ of these pseudoquadrangles. Each of these connected components is incident to exactly 2 obstacles and induces exactly one edge of the weak visibility graph. Therefore the weak visibility graph is a tree. □

Proof of Theorem 3. Omitted for lack of space. □

4 Extension to non-convex obstacles

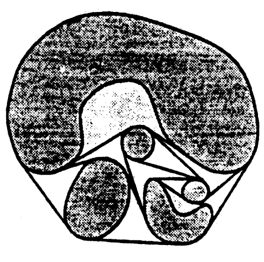

Figure 3: Pseudo–triangulation of a configuration of 5 obstacles.

In this section we extend our analysis to configurations of not necessarily convex *obstacles*. An *obstacle* is the interior of an injective smooth closed regular curve. Let $O = \{O_1, \ldots, O_n\}$ be a family of n disjoint obstacles. We denote by C_0 the convex hull of the family of obstacles and by C_i the relative convex hull of O_i with respect to the collection of obstacles, i.e., the *interior* of the shortest curve (in the closure of $\mathcal{R}^2 \setminus \cup_{i=1}^n O_i$) homotopy equivalent to the boundary of O_i (this shortest curve is not necessarily injective; its interior can be defined as the set of points in the plane whose winding number with respect to the curve is equal to $+1$ [5]). The complement in $\mathcal{R}^2$ of the union of the C_i ($i \geq 1$) is called *free* space; the union $\cup_{i=1}^n (C_i \setminus O_i)$ is called *semi-free* space. We denote by h_i ($i \geq 0$) the number of bitangents (counting multiplicities) on the boundary of C_i and by l_i ($i \geq 1$) the number of connected components of $C_i \setminus O_i$. Set $l(O) = \sum_{i=1}^n l_i$, and $h(O) = \sum_{i=0}^n h_i$. The set $C_0 \setminus \cup_{i=1}^n C_i$ is called the relative convex hull of the family of obstacles. We denote by $h'(O)$ the number of bitangents lying on the *boundary* of the relative convex hull. Observe that $h(O) = h'(O) + \omega(O)$, where $\omega(O)$ is the number of free bitangents incident on both sides upon semi-free space (these bitangents are counted twice in $h(O)$). As in the case of convex obstacles we define a pseudo–triangulation to be a subdivision of the plane induced by the obstacles and a maximal family of pairwise non-crossing free bitangents (see Figure 3 for an illustration of these notions). Lemmas 1, 2 and 4 have their counterparts in the non-convex case, and lead to the following generalization of Theorem 2.

Theorem 4 *Consider a collection O of n pairwise disjoint obstacles. Then the number of free bitangents of the collection is at least $6n - 6 + 2l(O) - h(O) + \omega(O)$. Furthermore the following assertions are equivalent.*

1. *its number of free bitangents is minimal (i.e., $4n - 4$).*
2. *the size of its relative convex hull is maximal (i.e., $h(O) = h'(O) = 2(n + l(O) - 1)$).* □

Next we consider the problem of *characterizing* the minimal visibility types. Observe that a necessary condition for minimality ($4n-4$) of the number of free bitangents is the absence of semi-free bitangents (i.e. bitangents with at least one endpoint lying in semi-free space). A configuration which satisfies this condition is called *regular*. Our first step is to represent a minimal visibility type by a map (= plane digraph) augmented with a 3-coloring of the set of edges. According to the previous theorem the tangent visibility graph of a regular configuration O is minimal iff. its relative convex hull is maximal or, equivalently (see the proof of the previous theorem), iff. bounded free space has exactly $n - 1 + l(O)$ connected components, $2l(O)$ of which are pseudotriangles, while the remaining $n-1-l(O)$ are pseudoquadrangles (i.e., the union of two adjacent pseudotriangles). Each of these components is adjacent to exactly 2 obstacles. We construct a 3-colored edges map, denoted $G(O)$, as follows.

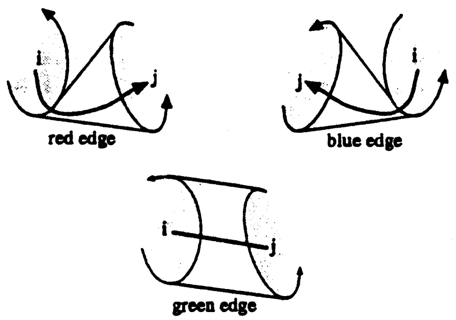

Figure 4: The three types of edges of the graph G.

1. Its set of nodes is $\{1, \ldots, n\}$ where i is a point lying inside the obstacle O_i.
2. If O_i and O_j are adjacent along a pseudo-quadrangle Q then we connect i and j by an *undirected green edge*, lying in $O_i \cup O_j \cup Q$.
3. If O_i and O_j are adjacent along a pseudo-triangle T, then the canonical labels of the two bitangents in the boundary of the pseudotriangle are (i, j) and $(i, -j)$, or (j, i) and $(-j, i)$ (up to a permutation of i and j and reorientation of the bitangents); in the former case we connect i and j by a *red directed edge*, in the latter case by a *blue directed edge* from i to j (see Figure 4).

Lemma 5 *The visibility cycle $C(O)$ of a configuration of obstacles O determines the map $G(O)$, and conversely.* □

It remains to characterize the maps $G(O)$ when O ranges over the set of configurations with minimal visibility type; we denote the set of these maps by $\mathcal{G}$. One can show that $\mathcal{G}$ is stable by contraction of a green edge or a digon face; furthermore any element in $\mathcal{G}$ can be reduced to a single node by a sequence of contractions of green edges and digons. However $\mathcal{G}$ is not the set of maps reducible to a single node by contractions of green edges and digons. It is a proper subset that we define now. The set of *admissible maps* $\mathcal{G}'$ is the smallest set defined as follows, see also Figure 5.

1. The map consisting of a single node belongs to $\mathcal{G}'$.
2. If G_1 and G_2 are disjoint maps in $\mathcal{G}'$, let G be the map obtained by connecting a node on the unbounded face of G_1 and a node on the unbounded face of G_2 by an undirected green edge. Then $G \in \mathcal{G}'$.
3. If G_1 and G_2 are disjoint maps in $\mathcal{G}'$, let G be the map obtained by connecting a node, say z, on the unbounded face of G_1 to one or two nodes, say t and t', on the unbounded face of G_2 by two colored directed edges, one red and one blue, such that the orientation of the red edge is consistent with the counterclockwise orientation of the boundary of the unbounded face of G. Then $G \in \mathcal{G}'$.

Observe that the coloring of the edges is superfluous since it can be deduced from the embedding (or alternatively the orientation of the edges is superfluous since it can be deduced from the coloring of the edges). There are 12 unlabeled admissible maps on 3 nodes depicted in Figure 6. By recurrence we can easily shown that that any admissible map G on n nodes is the map $G(O)$ of some collection O of n obstacles; the converse is also true.

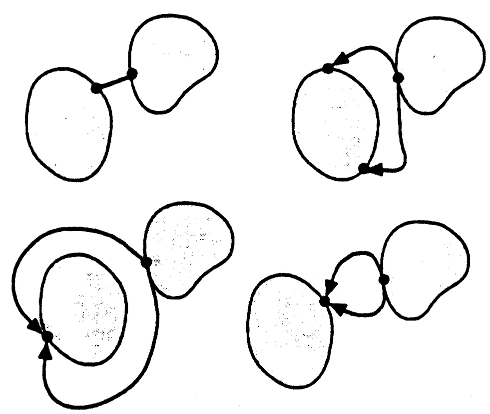

Figure 5: The two operations to construct admissible maps.

Theorem 5 *The set of minimal visibility types on n obstacles is in 1-1 correspondence with the set of admissible maps.* □

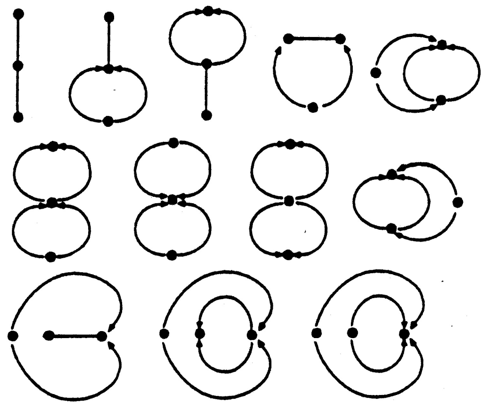

Figure 6: The 12 admissible unlabeled maps on 3 nodes.

5 Conclusion

We have proven that $4n-4$ is a tight lower bound for the size of tangent visibility graphs on n obstacles. We have also given a simple characterization of the corresponding minimal tangent visibility graphs (more precisely of the corresponding visibility types). Our main tool is the notion of pseudo–triangulation. It is expected that a better understanding of this notion will give insights in the characterization problem of tangent visibility graphs.

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