Level Difference of the Reconstructed Arrangement
(Extended Abstract)

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Abstract

In this paper we consider the problem about level difference between an original arrangement and arrangements reconstructed from partial information of the original arrangement, motivated from the learning theory where reconstruction of the arrangement from partial information can be considered as the inference of a network of threshold functions. If the difference of the level for an arbitrary point in the space between the original arrangement and the reconstructed arrangements is small, then even if each hyperplane may be different from the original one, the reconstructed arrangements consistent with partial information can be used as a good approximation of the original.

We prove that, given the level information of one point per cell for an unknown simple arrangement of finite set of hyperplanes in the d-dimensional space, the difference of the level of an arbitrary point between the original arrangement and any reconstructed arrangement, which is consistent with the level information, is (1) less than or equal to d for general d if the dual graph of the reconstructed arrangement is same as that of the original arrangement, and (2) less than or equal to 3 for planar case even if the dual graph of the reconstructed arrangement is different from that of the original arrangement.

1 Introduction

Using partial information of an unknown arrangement, in the d-dimensional space, how well can we reconstruct the arrangement? This problem is motivated from learning theory as described in the following. In the hidden layer of simple 3 layered neural networks, there are n threshold functions of d inputs and each function passes the value 1 or 0 to the output layer. In the output layer there is one threshold function which sums up all n inputs passed with multiplying weights, compare the summation with its threshold value, and returns the value 1 or 0 as the output of this neural network. Here n threshold functions of d inputs in the hidden layer naturally induces the arrangement $\mathcal{A}(H)$ of the set H of n hyperplanes in the d-dimensional space, and the threshold function of the output layer can be considered as the level of the cell of the arrangement $\mathcal{A}(H)$ when its weights are simply 1. To infer the threshold functions of such a neural network from sample point information can be considered as the reconstruction of an arrangement from partial information. If the largest difference of level of an arbitrary point in the space between the original arrangement and the reconstructed arrangement is bounded by a small constant, then even if each threshold function, that is, each hyperplane, inferred from the partial information may be completely different from the original one, the whole of the threshold functions, that is, the arrangement, consistent with partial information can be used as a good approximation for the threshold functions.

In this paper, we consider about the arrangement of n hyperplanes and as the partial information we use the level of points at least one point per each cell. Before stating our results, we first define terminology.

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Definition of level, level information and error: Let $\mathcal{A}(H)$ be the arrangement of a set H of n hyperplanes in R^d. Without loss of generality, we use the d-th axis of R^d for determining high and low.

(level) The level $L(c)$ of the cell c is the number of hyperplanes beyond c in the arrangement. The *level* $l(p, \mathcal{A}(H))$ of the point p in the arrangement $\mathcal{A}(H)$ is $L(c)$ if the point p belongs to the cell c of the arrangement $\mathcal{A}(H)$.

(level information) Given an arrangement $\mathcal{A}(H)$ in the d-dimensional space, *level information* I of $\mathcal{A}(H)$ is a finite set of points in the space, such that one point is selected from each cell of the arrangement, and each point p is equipped with its level $l(p, \mathcal{A}(H))$. Later, we slightly extend this "one point per each cell" to "at least one point per each cell".

(error) Let $\mathcal{A}(H)$ and $\mathcal{A}(H')$ be two arrangements. The error $e(p, \mathcal{A}(H), \mathcal{A}(H'))$ of a point $p \in R^d$ between two arrangements $\mathcal{A}(H)$ and $\mathcal{A}(H')$ is defined to be $|l(p, \mathcal{A}(H)) - l(p, \mathcal{A}(H'))|$. And the *error* $E(\mathcal{A}(H), \mathcal{A}(H'))$ between two arrangements $\mathcal{A}(H)$ and $\mathcal{A}(H')$ is defined to be $\max(e(p, \mathcal{A}(H), \mathcal{A}(H')))$ over all point p in R^d.

(consistency) Let I be the level information and $\mathcal{A}(H')$ be an arbitrary arrangement. If every cell of $\mathcal{A}(H')$ includes a point $p \in I$ and level information of p is same as $l(p, \mathcal{A}(H'))$, the arrangement $\mathcal{A}(H')$ is said to be *consistent* with the level information I.

Definition of the problem: Here, we define the problem rigorously. Let $\mathcal{A}(H)$ be an arrangement of n hyperplanes in the d-dimensional space, let I be the level information of $\mathcal{A}(H)$, and let $\mathcal{A}(H')$ be a reconstructed arrangement consistent with I. Then what is the error $E(\mathcal{A}(H), \mathcal{A}(H'))$ between $\mathcal{A}(H)$ and $\mathcal{A}(H')$? From now on, we use E instead of $E(\mathcal{A}(H), \mathcal{A}(H'))$ for simplicity.

Our results: We show that, if $\mathcal{A}(H)$ is simple, the error E is less than or equal to d if the dual graph of $\mathcal{A}(H)$ and $\mathcal{A}(H')$ are same, and less than or equal to 3 for planar case even if the dual graph of $\mathcal{A}(H')$ is different from that of $\mathcal{A}(H)$ (there are arrangements which are consistent with a given level information and whose dual graphs are different; see Aoki[1])

To prove the latter, we use the tool *level-zigzag* instead of dual graph and this level-zigzag can be created only from the level information I. We also show the examples of $E = n$ for the case that $\mathcal{A}(H)$ is not simple, and study the relationship between the error E and configuration of the arrangement.

Specifically, in section 2, we consider about the relationship between dual graph and level information in the general d-dimensional space. From section 3 we concentrate on the planar case. In section 3, we introduce the tool *level-zigzag* and *outer-line* created from the level information I. We treat inside of the outer-line and outside of the outer-line separately in section 4 and section 5. Here we show that if the arrangement is simple, the error is less than or equal to 3 inside the outer-line and less than or equal to 1 outside the outer-line. As for the non-simple case, if the arrangement has parallel lines, the error is less than or equal to $m + 1$ outside of the outer-line and 3 inside, where m is the maximum number of parallel lines. If more than two lines intersect at one point, the error is less than or equal to $m + 1$ and 1 outside, where m is the maximum number of lines intersecting at one point. And these are easily extended for the combined case.

2 Dual graph and level information

In this section we consider about the relationship between the sample point set and dual graph of the arrangement.

Let $\mathcal{A}(H)$ be an arrangement of n hyperplanes in the d-dimensional space. A *dual graph* of an arrangement $\mathcal{A}(H)$ is the graph whose vertex set consists of cells of the arrangement and whose edge set contains an edge c_1c_2 if and only if the cells c_1 and c_2 are adjacent in the arrangement.

Observation 1 *When each cell of $\mathcal{A}(H)$ has one and only one sample point, it is possible to embed the dual graph $\mathcal{G}(\mathcal{A}(H))$ of $\mathcal{A}(H)$ in such a way that each vertex of $\mathcal{G}(\mathcal{A}(H))$ is on the sample point*

of the corresponding cell. And, this graph can be straight line embedding in the planar case, and straight k-face embedding for general R^d.
Let $\mathcal{G}(\mathcal{A}(H))$ be the straight embedded dual graph of $\mathcal{A}(H)$ as above. Then we can observe the following.

Observation 2

- *Every cell of $\mathcal{G}(\mathcal{A}(H))$ forms d cubic shape, but not necessarily convex. Hence, every cell of $\mathcal{G}(\mathcal{A}(H))$ has 2d facets and 2^d vertices.*
- *Inside a cell of $\mathcal{G}(\mathcal{A}(H))$, there exists one and only one intersection point, that is, 0-face of $\mathcal{A}(H)$.*
- *If $\mathcal{A}(H)$ is simple, then each closed cell of $\mathcal{G}(\mathcal{A}(H))$ has $\binom{d}{i}$ vertices of level $k+i$, for $0 \leq i \leq d$, where k is the minimum level of vertex of corresponding cell of $\mathcal{G}(\mathcal{A}(H))$, and $0 \leq k \leq n-d$.*
- *A hyperplane of the original arrangement $\mathcal{A}(H)$ crosses the pairs of opposite facet facing to each other, so that every hyperplane divides one cubic cell into two cubic cells.*

From observation 2 and the fact that every 0-face of $\mathcal{A}(H)$ can only move inside the cell of $\mathcal{G}(\mathcal{A}(H))$ since all hyperplanes of $\mathcal{A}(H)$ is linear, we obtain the following lemma.

Lemma 1 *$k \leq l(p) \leq k+d$ for any point p in a cell of dual graph $\mathcal{G}(\mathcal{A}(H))$ which has vertices from level k to level $k+d$.*

This immediately implies the following.

Lemma 2 *E is less than or equal to d inside the closed cells of $\mathcal{G}(\mathcal{A}(H))$.*

There is a single open cell in $\mathcal{G}(\mathcal{A}(H))$. For this cell we obtain the following.

Lemma 3 *If $\mathcal{A}(H)$ is simple, $E \leq d-1$ for the open cell of the dual graph $\mathcal{G}(\mathcal{A}(H))$.*

Proof is omitted due to the space limitations.

Next, we consider about the case that $\mathcal{A}(H)$ is not simple, obviously upper bound of E is n from its definition, and we can construct two types of $E = n$ example as follows.

1. If all n hyperplanes of $\mathcal{A}(H)$ are parallel, then $\mathcal{G}(\mathcal{A}(H))$ becomes zigzag line having no cell, and, in such case, at infinity, E is n and this attains the upper bound.
2. If all n hyperplanes of $\mathcal{A}(H)$ intersect at one point p_1 and all hyperplanes of $\mathcal{A}(H')$ intersect at a point p_2 which is located a little below p_1, then the error $e(p, \mathcal{A}(H), \mathcal{A}(H'))$ on the point p between p_1 and p_2 is n, hence, E is n.

Here we obtain the following main theorem of this section.

Theorem 1 *Suppose the dual graph $\mathcal{G}(\mathcal{A}(H))$ of the original arrangement $\mathcal{A}(H)$ is same as the dual graph $\mathcal{G}(\mathcal{A}(H'))$ of the reconstructed arrangement $\mathcal{A}(H')$ which is consistent with the level information I of $\mathcal{A}(H)$. If $\mathcal{A}(H)$ is simple then E is less than or equal to d, and if $\mathcal{A}(H)$ is not simple then $E=n$ for the worst case, and these are tight.*

3 Definition of Level-zigzag and Outer-line

In this section, we introduce the tool *level-zigzag* and *outer-line* of the sample point information in R^2, which are intended to play a part of dual graph implicitly. Without loss of generality, we consider that R^2 is the xy-plane, y is used for judging upper or lower, and x is the only remaining axis.

Definition of level-zigzag and outer-line: Let I be a level information of $\mathcal{A}(H)$.

(level-zigzag) *k-level-zigzag* is defined as a zigzag line which connects all k level points of I with increasing order of x(Figure 1).

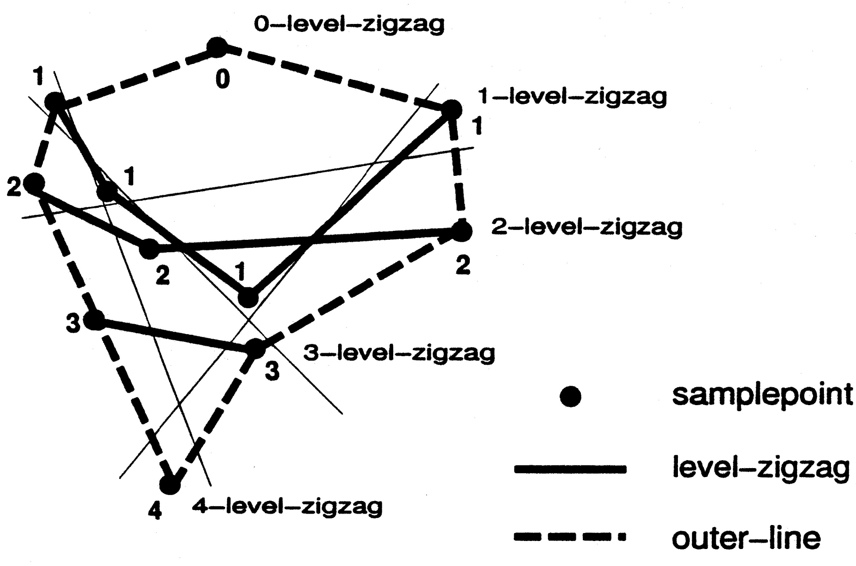

Figure 1: Level-zigzag and outer-line

Then, we have $n+1$ k-level-zigzags for $0 \leq k \leq n$, including one point of 0-level-zigzag and n-level-zigzag. Except 0-level-zigzag and n-level-zigzag, k-level-zigzag has at least 2 endpoints. Let p_{lk} denote the left endpoint of k-level-zigzag and p_{rk} denote the right endpoint of k-level-zigzag for $0 \leq k \leq n$. Both p_{l0} and p_{r0} (p_{ln} and p_{rn}, respectively) denote the same point.

(outer-line) *Outer-line* is defined as the polygon whose edge is the line between the endpoints p_{lk} and $p_{l(k+1)}$, and between p_{rk} and $p_{r(k+1)}$ for $0 \leq i \leq n-1$(Figure 1).

Lemma 4 *Outer-line forms a simple polygon.*

Proof: Let I be the level information. From section 2, there is the straight line embedding of the dual graph $\mathcal{G}(\mathcal{A}(H))$, and the outer-line is exactly same as the outer-line of embedding. The dual graph $\mathcal{G}(\mathcal{A}(H))$ of the arrangement on R^2 is planar graph, and the lemma follows. □

The region inside the outer-line can be considered as the collection of the closed cells of the dual graph $\mathcal{G}(\mathcal{A}(H))$ of the arrangement $\mathcal{A}(H)$ in section 2. And the region outside the outer-line can be considered as a single open cell of $\mathcal{G}(\mathcal{A}(H))$ of $\mathcal{A}(H)$, respectively. In the sequel, we consider inside and outside of the outer-line separately.

4 Inside the Outer-line

In this section we consider about the level-zigzag and estimate error E inside the outer-line.

In the arrangement $\mathcal{A}(H)$, if cells c_1 and c_2 is in the same level, $L(c_1) = L(c_2)$, and c_1 and c_2 have a common 0-face, cells c_1 and c_2 are called ***neighbor cells*** in $\mathcal{A}(H)$, and this 0-face is called ***inter-neighbor point*** of neighbor cells.

Observation 3 *k-level-zigzag is composed by straight line segments such that each segment has two endpoints in two neighbor cells from left to right.*

From this observation, we obtain the following.

Lemma 5 *The level of a point p on k-level-zigzag is $k-1 \leq l(p, \mathcal{A}(H')) \leq k+1$ for any arrangement $\mathcal{A}(H')$ that is consistent with the level information I of the original arrangement $\mathcal{A}(H)$.*

Proof: Let c_1 and c_2 be two neighbor cells such that $L(c_1) = L(c_2) = k$, and let p_1 (p_2, respectively) be the point of level information I corresponding to c_1 (c_2, respectively). There exists an inter-neighbor point z of the cells c_1 and c_2. Let z_l be a point located a little lower than z, then $l(z_l, \mathcal{A}(H)) = k+1$.

Suppose there exists a point x on the segment p_1p_2 such that level $l(x, \mathcal{A}(H))$ is greater or equal to $k+2$. Then between z_l and x, there exists at least one line l_x, such that z_l is above l_x, x is below l_x, and both p_1 and p_2 are above l_x. This contradicts the assumption that p_1p_2 is a straight line segment. With similar way we can prove $l(x, \mathcal{A}(H)) \geq k-1$. □

From this we can obtain the following.

Lemma 6 *k-level-zigzag never intersects with other k′-level-zigzag, with $k' \leq k-2$ or $k' \geq k+2$.*

Lemma 7 *$l(p, \mathcal{A}(H')) \leq k+1$ for any point p above or on k-level-zigzag, and $l(p, \mathcal{A}(H')) \geq k-1$ for any point p below or on k-level-zigzag.*

If the reconstructed arrangement $\mathcal{A}(H')$ is consistent with I, then each cell of the dual graph $\mathcal{G}(\mathcal{A}(H'))$ is consisted from one point on k-level-zigzag, two points on $(k+1)$-level-zigzag and one point on $(k+2)$-level-zigzag. This implies the following.

Lemma 8 *Any cell c of $\mathcal{A}(H)$ with $L(c) = k$ does not intersect with either $(k-2)$-level-zigzag or $(k+2)$-level-zigzag.*

Here we observe about the relationship between level-zigzag and dual graph of $\mathcal{A}(H)$.

Observation 4 *Every line segment which is a component of level-zigzag can be considered as one of the diagonals of the cell of the straight line embedded dual graph $\mathcal{G}(\mathcal{A}(H))$ of $\mathcal{A}(H)$. In other word, every component segment k_1k_2 of k-level-zigzag has a corresponding pair of vertex k_u and k_l, such that k_u (k_l, respectively) is on the $(k-1)$-level-zigzag ($(k+1)$-level-zigzag, respectively), and segment k_1k_2 is incident to two triangles $\triangle k_1k_2k_u$ and $\triangle k_1k_2k_l$. And the square $\Box k_uk_1k_lk_2$, which is a union of two triangles above, can be considered as a cell of $\mathcal{G}(\mathcal{A}(H))$. Hence our result $E = 3$ can be considered to be caused by the different triangulation for the sample point set in the plane.*

This k-level-zigzag is easily extended for another type of level information such that sample point is selected at least one point per each cell and lemma 7 or 8 follows in such case. But proof is omitted by the space limitation.

Next we consider about the case that the arrangement $\mathcal{A}(H)$ is not simple. If the arrangement is not simple, the condition is either

1. some lines of $\mathcal{A}(H)$ are parallel to each other, or
2. more than 2 lines intersect at one point.

In the case that lines are parallel, all discussions in this section above can be fit for use. And if m lines intersect at one point p, the cell of $\mathcal{G}(\mathcal{A}(H))$ corresponding to p is not a square but a polygon which has $2m$ edges and vertices, such that one of the vertices is on the k-level-zigzag (on the $(k+m)$-level-zigzag, respectively) and two vertices on k'-level-zigzag for $k < k' < k+m$.

Thus we obtain the following main theorem of this section.

Theorem 2 *Inside the outer-line, if the arrangement $\mathcal{A}(H)$ is simple, or even if the arrangement $\mathcal{A}(H)$ and $\mathcal{A}(H')$ includes some parallel lines, the error E between the original arrangement $\mathcal{A}(H)$ and reconstructed arrangement $\mathcal{A}(H')$ consistent with the level information I is less than or equal to 3. If m lines intersect at one point, where $m > 2$, the error E is less than or equal to $m+1$ inside the outer-line.*

5 Outside the outer-line

In this section, we consider about the error E outside the outer-line. Here we use similar technique which we used at lemma 1 of section 2, but this time, not for a cell of the dual graph.

Observation 5 *Outside the outer-line, the lines are sorted by their slope, since no line intersects outside the outer-line. And one and only one line of H crosses the edge of this simple polygon of the outer-line.*

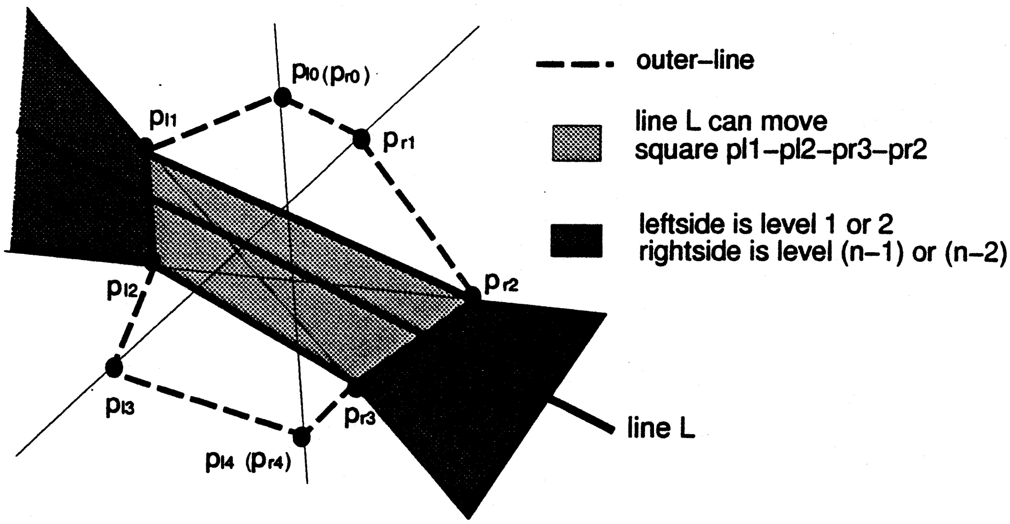

Figure 2: Outside of the outer-line

Let p_{lk} (p_{rk}, respectively) be the left (right, respectively) endpoint of k-level-zigzag. Let e_{lk} (e_{rk}, respectively) be the edge of outer-line, connecting p_{lk} and $p_{l(k+1)}$ (p_{rk} and $p_{r(k+1)}$) for $0 \leq i \leq n-1$. From the above observation, we can obtain the following.

Lemma 9 *The line l of H crosses two and only two edges of outer-line. If l crosses e_{lk}, then l also crosses $e_{r(n-k-1)}$.*

Lemma 10 *The line l of H can move only inside the square $\Box p_{lk}p_{l(k+1)}p_{r(n-k)}p_{r(n-k-1)}$* (Figure 2).

Here, we consider about the diagonals of these squares described above. Let d_k be the line through p_{lk} and $p_{r(n-k)}$. Using these diagonal lines, we divide the space outside of the outer-line. Let o_{lk} (o_{rk}, respectively) be an open region, surrounded by d_k, e_{lk}, d_{k+1} ($d_{n-k}, e_{rk}, d_{n-k-1}$, respectively).

Lemma 11 *The level of point p in o_{lk} or o_{rk} is k or $k+1$, for $0 \leq k \leq n-1$.*

Next we consider about the case that $\mathcal{A}(H)$ is not simple.

1. If more than two lines intersect at one point, we get same result for outside of outer-line.
2. If m lines are parallel, m lines can move inside the square $p_{li}, p_{l(i+m)}, p_{r(n-i)}, p_{r(n-i-m)}$.

Here we obtain the following main theorem of this section.

Theorem 3 *Outside of the outer-line, if the arrangement is simple, then the error E is less than or equal to 1. If the arrangement is not simple, it is enough to consider about the parallel lines. And if the arrangement has parallel lines, the error is less than or equal to $m+1$ outside of the outer-line, where m is the maximum number of parallel lines.*

6 Future work

To reconstruct an arrangement from the level information I, the time complexity of a naive algorithm is inevitably exponential. Hence an efficient algorithm is required for reconstructing an arrangement consistent with I. We may also consider about the error of the arrangements in higher dimensional space. Here the most different point from the planar case is that the level-zigzag facet cannot be uniquely determined from the level information.

References

[1] Y. Aoki: *The combinatorial complexity of reconstructing arrangements.* PhD thesis, Department of Information Science, University of Tokyo, 1993.